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DECOUPLING AND REDUCED ORDER MODELING OF TWO-TIME-SCALE CONTROL--ETC(U)

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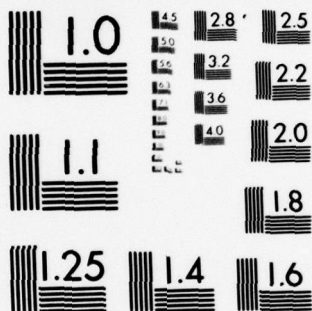
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DECOUPLING AND REDUCED ORDER MODELING  
OF TWO-TIME-SCALE CONTROL SYSTEMS

by

Leonard Robert Anderson

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## ABSTRACT

This study presents new analytical findings and a new computational method for time-scale decoupling and order reduction of linear control systems. A transformation of variables is considered which reduces the system to block-diagonal form with slow and fast modes decoupled. Once the decoupled form of the system is obtained, a small parameter can be identified which provides a measure of the system's time scale separation, and the methods of singular perturbations can be applied to yield reduced-order approximations of the original system. A particular reduced-order model so obtained overcomes certain difficulties encountered with other well known order-reduction methodologies.

The type of control system considered here is a very general, widely used form. The ability to identify a specific small parameter in all such systems provides a useful link to singular perturbations theory which requires a small parameter to be identified. Although the examples of two-time-scale systems presented here are relatively low order, due to recent advances in computational methods for matrices these decoupling techniques can be directly applied to large scale systems with orders of 100 or more.

In Chapter 1, a two-time-scale property is defined in terms of system eigenvalues, and a general class of linear transformations which yield time scale decoupling is also defined.

Chapter 2 considers two types of decoupling transformations, classical modal matrix transformations which decouple all modes of the system, and a new block-decoupling transformation introduced in a short 1975 paper by Petar V. Kokotovic which appeared in the IEEE Transactions on Automatic Control. Matrices  $L$  and  $K$  used in this transformation satisfy a nonsymmetric algebraic Riccati equation and a Lyapunov equation, respectively.

Application of the block-decoupling transformation to reduced order modeling is considered. Past work on reduced order modeling based on modal analysis is described and a new model based on a zeroth order outer solution is introduced. The differences in dynamic characteristics between new and old reduced order models are illustrated with a second-order example. Feedback matrix design using block decoupling transformations is described.

In Chapter 3, the nonsymmetric algebraic Riccati equation is studied and a basic theorem establishing new conditions for existence of solutions is presented. It is proven that for any given two-time-scale system the  $L$  and  $K$  matrices used in the decoupling transformation are real

and unique. The resulting decoupling transformation is always nonsingular.

Chapter 4 presents a new method of computing the  $L$  and  $K$  matrices. It is shown that the method has an asymptotic rate of convergence equal to the system's small parameter, and therefore converges rapidly for strongly two-time-scale systems. The method is initialized using certain of the system eigenvectors corresponding to either the system slow or fast modes, and then highly accurate  $L$  and  $K$  matrices can be obtained by linear iteration.

In Chapter 4, the choice of the order of the slow subsystem and the ordering of state variables is discussed. The relationship between ordering and the norms of the  $L$  and  $K$  matrices is described using specific slow and fast mode coupling ratios.

The stability and rate of convergence of the numerical decoupling method and the accuracy of the new reduced order model is demonstrated in Chapter 5 using two examples from recent literature. They are a fourth order model of F8 aircraft longitudinal dynamics and a sixteenth order model of the F100 turbofan engine.

The time-scale decoupling transformation is extended in Chapter 6 to time-varying linear systems. Methods of computing a time-varying  $LK$  transformation are presented and illustrated with a second-order example.

## CHAPTER 1

### INTRODUCTION

Physical systems often exhibit behavior that includes both fast and slow dynamic response, and it is common engineering practice to separately model and study the fast and slow phenomena. Such dynamic separation can simplify both analytical and computational tasks, and becomes a necessity for the study of large-scale (i.e., large order) systems, such as those encountered in structural dynamics (Foss, 1958), and power distribution systems (Van Ness, 1977). Past approaches to the separation of fast and slow dynamics have either relied upon engineering judgment to choose some variables as slow (and neglect the exact modal structure of the system), or employed modal variables which lack physical meaning and are computationally awkward. Here we present a new approach to the analysis of such systems which both clarifies the fast/slow dynamic separation task and overcomes common computational difficulties.

This study will be limited to dynamic systems which can be approximated by a set of linear first-order differential equations of the form

$$\frac{dx}{dt} = \dot{x} = Ax + B(t)u, \quad 0 \leq t \leq T \quad (1.1)$$

where  $x$  is an  $n$ -dimensional vector of state variables with  $x(0)$  often specified,  $u = u(t)$  is an  $m$ -dimensional vector of control variables generally specified on  $[0, T]$ , and  $A$  and  $B(t)$  are real matrices of appropriate dimensions. Such systems often include a  $p$ -dimensional vector of output variables  $w$ , which is a linear combination of the state and control

$$w = C(t)x + D(t)u, \quad (1.2)$$

$C$  and  $D$  being real. The matrices  $B$ ,  $C$ , and  $D$  are assumed to be continuous functions of time  $t$  on  $[0, T]$  and  $A$  is assumed to be constant. The systems (1.1), (1.2) will be referred to as a control system, and the case of control systems with time varying  $A$  matrix is considered in a later chapter. In applications, the variables  $x$ ,  $u$  and  $w$  will usually represent physical quantities.

As proposed by Chow and Kokotovic (1976), the control system will be called strongly two-time-scale or simply two-time-scale if the eigenvalues of  $A$ , represented as  $\lambda(A)$ , can be separated by absolute values into nonempty sets  $S$  and  $F$  with  $n_1$  elements  $s_i$  and  $n_2$  elements  $f_j$  respectively, such that

$$|s_i| \ll |f_j| \text{ for all } s_i \text{ in } S \text{ and } f_j \text{ in } F, \quad (1.3)$$

and  $n_1 + n_2 = n$ . If this relationship is only satisfied for the weak inequality

$$|s_i| < |f_j| \text{ for all } s_i \text{ in } S \text{ and } f_j \text{ in } F, \quad (1.4)$$

then the control system will be referred to as weakly two-time-scale. The special case of control systems which do not satisfy (1.4), i.e., for which all eigenvalues of  $A$  have equal modulus, will only be treated briefly in a later chapter. This algebraic definition of two-time-scale control systems does not necessarily imply a clearly identifiable slow/fast structure in the dynamic response of  $x(t)$  or  $w(t)$  which depend upon the specific control matrix  $B(t)$  and control input  $u(t)$ .

The reason for using the slow/fast terminology becomes clear if one considers the homogeneous problem with zero control. Likewise, for problems with slowly varying  $B(t)u(t)$  and for time-intervals longer than the fast mode time constant  $\frac{1}{\min |Re f_j|}$ , one can expect the dynamic response to often have a boundary layer structure at the endpoints corresponding to the fast eigenvalues  $f_j$  plus a slowly-varying part throughout  $[0, T]$  corresponding to the slow eigenvalues  $s_i$ .

For any two-time-scale system we can order the eigenvalues by modulus so that

$$|s_i| \leq |s_{i+1}|, \quad 1 \leq i \leq n_1 - 1$$

$$|f_j| \leq |f_{j+1}|, \quad 1 \leq j \leq n_2 - 1$$

then the ratio

$$\mu \equiv |s_{n_1}| / |f_1| \ll 1 \quad (1.5)$$

will define the system small parameter which provides a measure of the system's time scale separation and identifies the system as a singular perturbations problem. Such systems with slow and fast variables are also described in the literature as stiff (Flaherty and O'Malley 1977, and Willoughby 1974) or illconditioned (Kokotovic 1975). This particular choice of small parameter was also proposed by Kelley (1973).

Other additional constraints are often placed on the spectrum of the  $A$  matrix to insure stability, or insure that there are no highly oscillatory fast modes, but these constraints will be introduced later as they are needed. Since our initial purpose is only to identify the slowly varying and rapidly varying parts of system (1.1), the two-time-scale assumption will suffice.

It is possible for the spectrum of  $A$  to cluster by modulus into three or more groups so the system dynamics can be analyzed in three or more time scales. In such cases,

the system possesses two or more small parameters defined analogously to (1.5). Only the two-time-scale case will be considered here, although by repeated application of these time scale decoupling techniques the multiple time scale case could be treated.

Given a two-time-scale control system with  $n_1$  slow modes and  $n_2$  fast modes, we partition the state vector  $x$  into subvectors  $x_1$  and  $x_2$  of dimensions  $n_1$  and  $n_2$ , and partition the  $A$  and  $B$  matrices accordingly so that the control system (1.1) can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u. \quad (1.6)$$

Consider a nonsingular linear transformation of the state variables  $x$  to new dynamic variables  $\xi$  represented as

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \equiv \Gamma x.$$

The new control system defined by this transformation of variables is

$$\dot{\xi} = \theta \xi + \phi u \quad (1.7a)$$

$$\omega = \psi \xi + Du \quad (1.7b)$$

where  $\theta = \Gamma A \Gamma^{-1}$ ,  $\phi = \Gamma B$ , and  $\psi = C \Gamma^{-1}$ . If system (1.7) has the block-diagonal form

$$\dot{\xi}_1 = \theta_{11} \xi_1 + \phi_1 u \quad (1.8a)$$

$$\dot{\xi}_2 = \theta_{22} \xi_2 + \phi_2 u \quad (1.8b)$$

where  $\theta$  and  $\phi$  are partitioned compatibly with (1.6), there is no interaction between  $\xi_1$  and  $\xi_2$ , so we have achieved a decoupling of the system's variables. Since matrices  $A$  and  $\theta$  are similar, they have the same eigenvalues, i.e.,

$$\lambda(\theta) = \lambda(A).$$

If transformation  $\Gamma$  decouples system (1.1) and, in addition, the eigenvalues are partitioned as

$$\lambda(\theta_{11}) = S, \quad \lambda(\theta_{22}) = F, \quad (1.9)$$

then we have achieved a time scale decoupling of the

two-time-scale system. The subsystems (1.8a) and (1.8b) will represent the system's slow dynamic and fast dynamic components, respectively. Now partition  $\Gamma$  and  $\Gamma^{-1}$  as

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \quad \Gamma^{-1} = [\Delta_1 \ \Delta_2]$$

where  $\Gamma_1$ ,  $\Gamma_2$ ,  $\Delta_1$  and  $\Delta_2$  have dimensions  $n_1 \times n$ ,  $n_2 \times n$ ,  $n \times n_1$ , and  $n \times n_2$ . Given any initial state  $x(0)$  and control function  $u(t)$ , the solution to the initial value problem for the system (1.1) is given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B(\tau)u(\tau)d\tau.$$

Employing the block-diagonal structure of the transformed system (1.8), the solution  $x(t)$  can be defined as the direct sum of slow mode and fast mode constituents  $x_s$  and  $x_f$ , i.e.,

$$x(t) = x_s(t) + x_f(t)$$

where

$$x_s(t) \equiv \Delta_1 \{ e^{\theta_{11}t} \Gamma_1 x(0) + \int_0^t e^{\theta_{11}(t-\tau)} \Gamma_1 B(\tau) u(\tau) d\tau \} \quad (1.10)$$

$$x_f(t) \equiv \Delta_2 \{ e^{\theta_{22} t} \Gamma_2 x(0) + \int_0^t e^{\theta_{22}(t-\tau)} \Gamma_2 B(\tau) u(\tau) d\tau \}.$$

As long as (1.8) and (1.9) are satisfied, the slow mode and fast mode constituents  $x_s(t)$  and  $x_f(t)$  are unique on  $[0, T]$  for all  $\Gamma$ .

Although many different time scale decoupling transformations  $\Gamma$  are possible, two specific choices for which have especially suitable algebraic structure will be considered here. These two transformations are presented in Chapter 2.

## CHAPTER 2

### MODAL DECOUPLING AND REDUCED ORDER MODELING

One well established method of time scale decoupling is through the use of modal variables as described, for example, by Zadeh and Desoer (1963) or Porter and Crossley (1972). In order to simplify the review of this method, assume for the present that  $A$  is nondefective, i.e., that it can be put into diagonal form by a similarity transformation.

#### 2.1. A Review of Modal Variables

Let the  $A$  matrix be represented in the spectral form

$$A = MJQ. \quad (1.1)$$

Here  $J$  is a diagonal matrix whose diagonal elements are the eigenvalues in  $\lambda(A)$ ,  $M$  is a modal matrix whose columns are the corresponding (right) eigenvectors of  $A$ , and  $Q = M^{-1}$ . The spectral form of a general matrix  $A$  with possibly generalized eigenvectors will be considered in Chapter 3.

For the classical method of modal variables, the transformation  $\Gamma$  is given via its matrix  $Q$ , and modal variables are defined by

$$z \equiv Qx. \quad (1.2)$$

The original system (less output variables)

$$\dot{x} = Ax + Bu \quad (1.3)$$

is transformed by (1.2) to the familiar completely decoupled form

$$\dot{z} = Jz + QBu \quad (1.4)$$

or, equivalently, the  $n$  scalar equations

$$\dot{z}_i = \lambda_i z_i + q_i Bu, \quad i = 1, \dots, n$$

where  $z^T = (z_1, \dots, z_n)$ ,  $J = \text{diag}(\lambda_1, \dots, \lambda_n)$  and  $q_i$  is the  $i$ th row of matrix  $Q$ . This completely decoupled form achieves more separation of eigenvalues than is needed. That is, for the study of two-time-scale systems it is only necessary to separate slow and fast eigenvalues. Modal variables are also difficult to use in practice for the following reasons:

- 1) The physical insight which may be inherent in the original variables,  $x$ , may be lost with the modal variables,  $z$ .

- 2) Modal variables may be complex quantities while the physical variables,  $x$ , are most apt to be real.
- 3) The transformation  $Q$  is nonunique, so the modal variables,  $z$ , are nonunique.
- 4) The modal matrix  $M$  becomes ill-conditioned (cf. Stewart, 1973, p. 297) as the eigenvalues of  $A$  approach each other.
- 5) The accuracy of the modal variables obtained is limited by the accuracy of the numerical scheme used to calculate eigenvalues and eigenvectors.

Nonetheless, the modal form (1.4) is sometimes employed in practical control problems since it may yield useful information on the control structure. That is, the  $j$ th element of the row vector  $q_1 B$  described above provides a measure of the influence of the  $j$ th control variable upon the  $i$ th mode of the system. This yields insight for controller design.

## 2.2. The LK Decoupling Transformation

A new transformation useful for two-time-scale systems was introduced by Kokotovic (1975) and provides an alternative to the use of modal variables. This transformation, which will be referred to as the LK transformation, involves a two step reduction of the system matrix  $A$  first to block-triangular form and then to block-diagonal form. An

earlier form of this transformation appears in the singular perturbations literature, cf. Wasow (1976), O'Malley (1969), and Chang (1972).

Consider matrices  $L$  and  $K$  of dimensions  $n_2 \times n_1$  and  $n_1 \times n_2$  respectively which define new vectors  $y_1$  and  $y_2$  of dimensions  $n_1$  and  $n_2$ , as follows:

$$y_1 = x_1 + Ky_2$$

$$y_2 = Lx_1 + x_2,$$

or, in matrix form

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.1)$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I + KL & K \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2.2)$$

so

$$y = Tx \quad \text{where} \quad T \equiv \begin{bmatrix} I + KL & K \\ L & I \end{bmatrix}.$$

Here  $I$  represents the identity matrix of appropriate dimension ( $n_1 \times n_1$  or  $n_2 \times n_2$ ).

If the  $L$  matrix is a solution of the nonsymmetric algebraic Riccati equation (ARE)

$$LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0, \quad (2.3)$$

where the  $A_{ij}$  are submatrices of the  $A$  matrix [as defined in Chapter 1, equation (1.6) i.e., (I.1.6)] then the transformation (2.1) reduces the control system (1.3) to block-triangular form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ LB_1 + B_2 \end{bmatrix} u$$

where

$$\tilde{A}_{11} \equiv A_{11} - A_{12}L, \quad \tilde{A}_{12} \equiv A_{12} \quad \text{and} \quad \tilde{A}_{22} \equiv A_{22} + LA_{12}.$$

Furthermore, if the  $K$  matrix satisfies the Lyapunov

equation

$$K\hat{A}_{22} - \hat{A}_{11}K + A_{12} = 0, \quad (2.4)$$

then the transformation (2.2) decouples (1.3) yielding

$$\dot{y}_1 = \hat{A}_{11}y_1 + \hat{B}_1u \quad (2.5a)$$

$$\dot{y}_2 = \hat{A}_{22}y_2 + \hat{B}_2u, \quad (2.5b)$$

which in matrix form is block-diagonal, i.e.,

$$\dot{y} = \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} y + \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} u \equiv \hat{A}y + \hat{B}u$$

where  $\hat{B}_1 \equiv (I + KL)B_1 + KB_2$  and  $\hat{B}_2 \equiv LB_1 + B_2$ .

If, in addition to satisfying ARE the  $L$  matrix separates the eigenvalues as required by (I.1.9), that is,

$$\lambda(\hat{A}_{11}) = S, \quad \lambda(\hat{A}_{22}) = F$$

then the block-diagonal system (2.5a,b) is time scale decoupled. Comparing with equation (I.1.8), we find that the  $n_1$ th order system (2.5a) represents the system's

slow mode components and the  $n_2$ -th order system (2.5b) represents its fast mode part. In Chapter 3 we will show that such  $L$  and  $K$  matrices always exist for weakly two-time-scale systems and they are real and unique. Accurate numerical values for  $L$  and  $K$  can be obtained as shown in Chapter 4.

One attractive feature of the transformation of variables defined by (2.2) is that the inverse transformation is

$$x = \begin{bmatrix} I & -K \\ -L & I + LK \end{bmatrix} y \equiv T^{-1}y. \quad (2.6)$$

Since the inverse is explicitly determined by  $L$  and  $K$ , no numerical matrix inversion procedure need be used, and the inverse exists whenever  $T$  is defined. Although the physical interpretation of the modal variables  $z$  is obscure, the relationship between  $x$  and the decoupled variables  $y$  is relatively transparent. If we decompose the state variables into fast and slow parts  $x_s$  and  $x_f$  as defined by (I.1.10), we will obtain

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_{1s} \\ x_{2s} \end{bmatrix} + \begin{bmatrix} x_{1f} \\ x_{2f} \end{bmatrix} = x_s + x_f \quad (2.7)$$

and since  $x_1 = y_1 - Ky_2$ , it's apparent that  $x_{1s} = y_1$ . The variables  $y_1$  therefore have the same physical dimensions as  $x_1$ . Nearly the same analogy can be made between  $y_2$  and  $x_2$ , i.e.,  $x_2 = -Ly_1 + (I + LK)y_2$  implies  $x_{2f} \approx y_2$  whenever  $LK$  is of small norm.

To summarize, the  $LK$  transformation has several distinct advantages compared to the use of modal variables  $z$  and the modal transformation  $M$  described in the previous section, namely:

- (1) The physical insight inherent in the  $x$  variables is retained in the time-scale-decoupled variables  $y$  since  $x_{1s} = y_1$  and  $x_{2f} \approx y_2$ .
- (2) The  $LK$  transformation  $T$  is always real and unique and its inverse can be obtained explicitly in terms of  $L$  and  $K$  without using numerical matrix inversion.
- (3) Matrices  $L$  and  $K$  are obtained from algebraic equations (2.3) and (2.4), whereas  $M$  and  $Q$  have to be obtained by numerical eigenanalysis and matrix inversion.

The practicality of the  $LK$  transformation will be fully demonstrated in later chapters.

Once the appropriate  $L$  and  $K$  matrices have been obtained the integrated form of the time scale decoupled system (2.5) becomes

$$x(t) = T^{-1} \{ e^{\tilde{A}t} T x(0) + \int_0^t e^{\tilde{A}(t-\tau)} T B(\tau) u(\tau) d\tau \}$$

or, applying (2.7) and the expanded form for  $T$ :

$$x_s(t) \quad (2.8)$$

$$= \begin{bmatrix} \bar{I} \\ -L \end{bmatrix} \{ e^{\hat{A}_{11}t} [(I + KL) K] x(0) + \int_0^t e^{\hat{A}_{11}(t-\tau)} \hat{B}_1(\tau) u(\tau) d\tau \}$$

$$x_f(t) \quad (2.9)$$

$$= \begin{bmatrix} -K \\ I + LK \end{bmatrix} \{ e^{\hat{A}_{22}t} [L I] x(0) + \int_0^t e^{\hat{A}_{22}(t-\tau)} \hat{B}_2(\tau) u(\tau) d\tau \}.$$

No approximations have been made at this point so this solution  $x(t) = x_s(t) + x_f(t)$  is exact.

Without additional assumptions on  $x(0)$  and  $B(t)u(t)$ , the slow/fast terminology continues to be merely suggestive and  $x_s(t)$  and  $x_f(t)$  need not be clearly distinguishable as slow and fast.

The decoupled form (2.8), (2.9) may be advantageous for numerical integration since the slow and fast modes are integrated separately to be later recombined to obtain the state  $x$ . If  $n_1 \approx n_2$  then approximately half as many multiplication and addition operations are required to

evaluate derivatives at one time step for the block-diagonal matrix form (2.5) compared to the original (1.3).

One of the main applications of modal decoupling is the derivation of reduced order models which approximate the slow dynamic behavior of the original system (1.1). Here maximum benefit is achieved if  $n_1 \ll n_2$  so that the reduced order model is of considerably lower order than the original model.

### 2.3. The Reduced Order Models of Davison and Marshall

In order to be able to neglect fast mode behavior and have  $x(t) \approx x_s(t)$  for  $t \gg \frac{1}{|\bar{f}_1|}$ , we will need the following fast mode stability assumption:

$$|s_{n_1}| \ll -\text{Re}(f_j) \quad \text{for all } f_j \text{ in } F. \quad (3.1)$$

This assumption guarantees that the fast modes are all stable and well damped relative to the slow modes. Since it is possible for

$$|s_{n_1}| \ll |\text{Re}(f_j)| \ll |\text{Im}(f_j)|,$$

the fast modes could still be highly oscillatory. Under fast mode stability if we are to consider control system response for large times, then to insure bounded state  $x(t)$

the following slow mode stability assumption will also be needed:

$$\operatorname{Re}(s_i) < 0 \text{ for all } s_i \text{ in } S. \quad (3.2)$$

The two conditions (3.1) and (3.2) taken together will be referred to as the stability assumption. In this and the following section it is assumed that  $A$  satisfies the fast mode stability assumption.

Reduced order modeling plays a central role in the study of large scale (high order) systems. One order reduction technique based on modal analysis [suggested by Nicholson (1964) and presented by Davison (1966)] has been widely studied (Sandell et al. 1978). The Davison model will be derived using a notation which is more concise than the original and is similar to that of Sandell et al. (1978). Consider the spectral form of  $A$  (assumed nondefective)

$$A = MJQ$$

partitioned compatibly with system (I.1.6) as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.3)$$

where  $A_{11}$ ,  $M_{11}$ ,  $J_1$  and  $Q_{11}$  have dimensions  $n_1 \times n_1$ , etc. and

$$\lambda(J_1) = S, \quad \lambda(J_2) = F. \quad (3.4)$$

If the modal variables  $z$  are also partitioned into sub-vectors  $z_1$  and  $z_2$  of dimensions  $n_1$  and  $n_2$  then the system equations in modal form (1.4) can be written as slow part

$$\dot{z}_1 = J_1 z_1 + [Q_{11} \quad Q_{12}] Bu \quad (3.5a)$$

and fast part

$$\dot{z}_2 = J_2 z_2 + [Q_{21} \quad Q_{22}] Bu. \quad (3.5b)$$

The transformation from modal variables back to state variables can be written as

$$x_1 = M_{11} z_1 + M_{12} z_2 \quad (3.6)$$

$$x_2 = M_{21} z_1 + M_{22} z_2.$$

The Davison model is based on the approximation that  $z_2(t) = 0$  for all time, i.e., the fast modes can be

disregarded. This implies that both  $z_2(t)$  and the non-homogeneous term in (3.5b) are small, i.e.,  $z_2 \cong 0$  and  $[Q_{21} \ Q_{22}]Bu \cong 0$ . This may or may not be a reasonable approximation depending upon the particular problem. We define

$$x_1 \cong M_{11}z_1 \equiv \tilde{x}_1$$

and

(3.7)

$$x_2 \cong M_{21}z_1 = M_{21}M_{11}^{-1}\tilde{x}_1 \equiv \tilde{x}_2$$

assuming  $M_{11}^{-1}$  exists. Substituting for  $z_1$  in (3.5a) yields the Davison reduced order system

$$\dot{\tilde{x}}_1 = M_{11}J_1M_{11}^{-1}\tilde{x}_1 + M_{11}[Q_{11} \ Q_{12}]Bu(t). \quad (3.8)$$

The general solution of (3.8) can be written as

$$\begin{aligned} \tilde{x}(t) &= \begin{bmatrix} \tilde{x}_1(t) \\ \tilde{x}_2(t) \end{bmatrix} \\ &= \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \left\{ e^{J_1 t} M_{11}^{-1} \tilde{x}_1(0) + \int_0^t e^{J_1(t-\tau)} [Q_{11} \ Q_{12}] B(\tau) u(\tau) d\tau \right\} \end{aligned} \quad (3.9)$$

which is Davison's approximate solution for the original system (1.3). If the initial state  $\tilde{x}_1(0)$  is chosen as

$$\tilde{x}_1(0) = M_{11}[Q_{11} \quad Q_{12}]x(0)$$

then comparing with equation (I.1.10), it is apparent that  $\tilde{x} = x_s$ . However, Davison did not specify this value for  $\tilde{x}_1(0)$ , but assumed  $\tilde{x}_1(0) = x_1(0)$ .

As observed by Chidambara (1967) one objection to Davison's model (3.8) is that for the case of constant inputs the steady state values of  $x_1$  and  $x_2$  are both in error. That is, the model does not reproduce the steady state values of the original system.

Now assume that  $A$  is slow mode stable as well as fast mode stable. For constant input  $\hat{u}$  we obtain the steady state value of the original system as

$$x(\infty) \equiv -M \begin{bmatrix} J_1^{-1} & 0 \\ 0 & J_2^{-1} \end{bmatrix} QB(\infty) \hat{u} \quad (3.10)$$

In contrast, the corresponding steady state for Davison's model (3.8) and (3.9) is given by

$$\hat{\tilde{x}}(\infty) = - \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} J_1^{-1} [Q_{11} \quad Q_{12}] B(\infty) \hat{u} \quad (3.11)$$

indicating the discrepancy. (In (3.10) and (3.11), the coefficients are evaluated at infinity.) It is apparent that (3.11) is simply the slow mode part of (3.10) if (3.10) is written in expanded form, i.e.,

$$\hat{\tilde{x}}(\infty) = x_s(\infty). \quad (3.12)$$

A second reduced order modeling technique developed independently by Marshall (1966) and Chidambara (1967) is currently being used to reduce the order of large scale systems (cf., e.g., Skira and DeHoff 1977). This model can be derived from the original differential equation for the partitioned state  $x_1$ , namely

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u. \quad (3.13)$$

Using the inverse modal transformation  $Q$  and  $z = Qx$ , we express  $x_2$  in terms of  $x_1$  and  $z_2$  as

$$x_2 = -Q_{22}^{-1}Q_{21}x_1 + Q_{22}^{-1}z_2.$$

The invertability of  $Q_{22}$  follows from the assumed invertability of  $M_{11}$ . Now the approximation  $\dot{z}_2 \approx 0$  is made in (3.5b) yielding

$$z_2(t) \approx -J_2^{-1} [Q_{21} \quad Q_{22}] Bu(t).$$

Substituting for  $x_2$  and  $z_2$  in (3.13) yields Marshall's reduced order system

$$\begin{aligned} \dot{\tilde{x}}_1 &= (A_{11} - A_{12}Q_{22}^{-1}Q_{21})\tilde{x}_1 \\ &+ [B_1 - A_{12}Q_{22}^{-1}J_2^{-1}[Q_{21} \quad Q_{22}]Bu(t)], \end{aligned} \quad (3.14)$$

with  $\tilde{x}_1(0) = x_1(0)$ . Marshall's model (3.14) does not necessarily imply that  $J_2 z_2$  and  $[Q_{21} \quad Q_{22}]Bu$  are small, but only that there is no difference between them. This is the same assumption to be made below in defining the zeroth order outer solution of a corresponding singularly perturbed system.

Writing  $A_{11}$  and  $A_{12}$  in their spectral forms

$$A_{1j} = M_{11}J_1Q_{1j} + M_{12}J_2Q_{2j}, \quad j = 1, 2$$

and using the identity

$$Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} = M_{11}^{-1}$$

it follows that

$$(A_{11} - A_{12}Q_{22}^{-1}Q_{21}) = M_{11}J_1M_{11}^{-1}.$$

Comparing (3.8) and (3.14) shows a strong similarity between Davison's model and Marshall's, i.e., they have the same eigenvalues and eigenvectors. The coefficients of the control term in the two models are not equal, however, i.e.,

$$M_{11}[Q_{11} \quad Q_{12}]B \neq B_1 - A_{12}Q_{22}^{-1}J_2^{-1}[Q_{21} \quad Q_{22}]B$$

so the variable  $\tilde{x}_1$  in Marshall's model does not equal  $x_{1s}$ . The primary advantage of Marshall's model is that it does yield the correct steady state value

$$\tilde{x}_1(\infty) = x_1(\infty) \quad (3.15)$$

since

$$\tilde{x}_1(\infty) = -M_{11}J_1^{-1}M_{11}^{-1}[B_1 - A_{12}Q_{22}^{-1}J_2^{-1}[Q_{21} \quad Q_{22}]B]\hat{u},$$

and after considerable manipulation it can be shown that

$$\hat{x}_1(\infty) = [M_{11} \quad M_{12}] \begin{bmatrix} J_1^{-1} & 0 \\ 0 & J_2^{-1} \end{bmatrix} Q \hat{B} u$$

which agrees with (3.10). (Coefficients are again evaluated at infinity.)

Another increasingly used approach to reduced order modeling of control systems is the reduced order approximation of singular perturbation theory. In this method, it is assumed that a small parameter  $\epsilon$  has already been identified, and the control system can be written in the form

$$\dot{x}_1 = A'_{11}x_1 + A'_{12}x_2 + B'_1u \quad (3.16a)$$

$$\epsilon \dot{x}_2 = A'_{21}x_1 + A'_{22}x_2 + B'_2u. \quad (3.16b)$$

Here, the prime notation is used to indicate that a different partitioning of the original system (1.3) may be chosen. The value  $\epsilon$  must be small relative to one and all matrices in (3.16) are usually considered to be the same order of magnitude in some suitable norm. The reduced order approximation to (3.16) obtained by setting  $\epsilon = 0$  and assuming  $A'_{22}$  invertible is

$$\dot{x}_1 = (A'_{11} - A'_{12}A'^{-1}_{22}A'_{21})x_1 + (B'_1 - A'_{12}A'^{-1}_{22}B'_2)u \quad (3.17a)$$

$$x_2(t) = -A_{22}'^{-1}(A_{21}'x_1(t) + B_2'u(t)). \quad (3.17b)$$

In order for the model (3.17) to yield a good approximation to (3.16) outside boundary layer regions, appropriate stability hypotheses are required (cf. Kokotovic, O'Malley and Sannuti 1976), like the fast-mode stability assumption of the previous section. One of the key advantages of this approach is that, if the system can be written in the form (3.16) then, even for systems of large order, the reduced order approximation (3.17) is easily obtained. Equations in form (3.16) are usually obtained from an analytical derivation where  $\epsilon$  may represent specific physical parameters or ratios of physical parameters, though such scaling is usually for low order systems. In some studies of large scale problems where the system equations in form (1.3) are obtained through numerical methods, the partition of  $x$  into  $x_1$ ,  $x_2$  is based on engineering judgment and the reduced order model (3.17) is obtained without considering the actual value of  $\epsilon$  (cf. Kelley 1973; and others). Harvey and Pope (1976) employ the above method (termed by them "residualization") to reduce a high order model of aircraft longitudinal dynamics for the purpose of evaluating control synthesis techniques.

#### 2.4. A New Reduced Order Model from Singular Perturbations

The singular perturbation method can be readily applied to the two-time-scale control system in block-diagonal form, namely

$$\dot{y}_1 = \tilde{A}_{11}y_1 + \tilde{B}_1u \quad (4.1a)$$

$$\dot{y}_2 = \tilde{A}_{22}y_2 + \tilde{B}_2u. \quad (4.1b)$$

In order to clearly describe the singular perturbation nature of (4.1), introduce the new small parameter

$$\sigma \equiv \frac{\|\tilde{A}_{11}\|}{\|\tilde{A}_{22}\|}.$$

Given any small number  $\delta > 0$ , the norm  $\|\cdot\|$  can be chosen so that

$$\|\tilde{A}_{11}\| \leq |s_{n_1}| + \delta |f_{n_2}|$$

and for any computable norm

$$|f_1| \leq |f_{n_2}| \leq \|\tilde{A}_{22}\|$$

(cf. Stewart 1973, p. 184, 270) so that  $\sigma$  will generally be smaller than  $\mu$ , cf (I.1.5), i.e.,

$$\sigma \leq \frac{|s_{n1}|}{|f_{n2}|} + \delta \leq \mu + \delta \ll 1.$$

Multiplying (4.1b) through by either small parameter  $\sigma$  or  $\mu$  clearly shows its singularly perturbed nature. Indeed, if one is to consider a linear control system as a singular perturbations problem, then with no loss of generality the system may be considered in the form

$$\dot{y}_1 = \hat{A}_{11}y_1 + \hat{B}_1u \quad (4.2a)$$

$$\mu\dot{y}_2 = \hat{A}_{22}y_2 + \hat{B}_2u \quad (4.2b)$$

where  $\hat{A}_{22} \equiv \mu\tilde{A}_{22}$  and  $\hat{B}_2 \equiv \mu\tilde{B}_2$  with no ambiguity as to the value of  $\mu$ ,  $\tilde{A}$  or  $\tilde{B}$ . Note that this rescaling makes the entries in  $\hat{A}_{22}$  of order  $O(1)$ .

The singular perturbation procedure eliminates the derivative term in the limit  $\mu \rightarrow 0$  because that term is multiplied by the specific small parameter  $\mu$ . In its physical interpretation, the derivative term is omitted because the dynamics of  $y_2(t)$  are fast and the derivative  $\dot{y}_2(t)$  is small after a typical fast dynamic transient has died out. Equation (4.2) illustrates the equivalence of these two approaches.

The following new reduced order model approximating the system's slow dynamics is obtained by setting  $\mu = 0$

in (4.2b):

$$\dot{y}_1 = \hat{A}_{11}y_1 + \hat{B}_1u \quad (4.3a)$$

$$\hat{y}_2 \equiv -\hat{A}_{22}^{-1}\hat{B}_2u \quad (4.3b)$$

$$\hat{x}(t) \equiv \begin{bmatrix} I & -K \\ -L & I + LK \end{bmatrix} \begin{bmatrix} \bar{y}_1(t) \\ \hat{y}_2(t) \end{bmatrix}. \quad (4.3c)$$

(The last notation corresponds to the new reduced order model.) For initial value problems, the initial conditions appropriate for (4.3a) are

$$y_1(0) = (I + KL)x_1(0) + Kx_2(0). \quad (4.3d)$$

(If our system were fast mode unstable, we could analogously specify terminal values for the state vector and require that the control component  $B(t)u(t)$  on  $[0, T]$  not excite these unstable modes. The more difficult problem when  $\hat{A}_{22}$  is conditionally stable deserves further study with both boundary values  $x(0)$  and  $x(T)$  and control term  $B(t)u(t)$  appropriately restricted [O'Malley 1969; O'Malley and Anderson 1979].) Comparing this model with the original decomposition of  $x$  into slow and fast components

$x = x_s + x_f$  we have an exact representation of the slow component, i.e.,

$$x_s(t) = \begin{bmatrix} I \\ -L \end{bmatrix} y_1(t) = \begin{bmatrix} I \\ -L \end{bmatrix} \{ e^{\tilde{A}_{11}t} [(I + KL)x_1(0) + Kx_2(0)] + \int_0^t e^{\tilde{A}_{11}(t-\tau)} [I + KL \quad K] B(\tau) u(\tau) d\tau \}. \quad (4.4)$$

Assuming the system (1.3) is also slow mode stable, the steady state response of (4.3) for constant input  $\hat{u}$  is

$$\begin{aligned} \hat{x}(\infty) &\equiv \begin{bmatrix} I & -K \\ -L & I + LK \end{bmatrix} \begin{bmatrix} -\tilde{A}_{11}^{-1} & 0 \\ 0 & -\tilde{A}_{22}^{-1} \end{bmatrix} \begin{bmatrix} I + KL & K \\ L & I \end{bmatrix} \hat{B}u \\ &= -T^{-1} \tilde{A}^{-1} T \hat{B}u \end{aligned} \quad (4.5)$$

(with evaluation at infinity). Comparing this form with the steady state response of (1.3), i.e.,

$$x(\infty) = -A^{-1} B(\infty) \hat{u},$$

the equivalence of the steady state responses

$$x(\infty) = \hat{x}(\infty)$$

is shown by the relationship between the original system and the block diagonal form, i.e.,

$$A = T^{-1} \hat{A} T.$$

Thus the new reduced order model (4.3) reproduces the slow mode component of the system response to both initial conditions and control inputs, and yields the correct steady state response to constant inputs overcoming both difficulties encountered in the Davison and Marshall models.

If the original control system includes output variables  $w$ , i.e.,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ w &= Cx + Du, \end{aligned} \tag{4.6}$$

then the new reduced order model may be written in a similar compact form. If a new output matrix  $\hat{C}$  is defined as

$$\hat{C} \equiv CT^{-1}$$

and partitioned as

$$\hat{C}_Y = [\hat{C}_1 \quad \hat{C}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

then the new model (4.3) may be written as

$$\begin{aligned} \dot{y}_1 &= \hat{A}_{11}y_1 + \hat{B}_1u \\ w &= \hat{C}_1y_1 + \hat{D}u \end{aligned} \tag{4.7}$$

where  $\hat{D} \equiv D - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2$  with initial condition  $y_1(0)$  specified by (4.3d).

### 2.5. A Second-Order Example

The differences in the accuracy of approximation between Davison's model, Marshall's model and the new model (4.3) can be illustrated by the following simple second-order example used by Chidambara (1967) and Wilson, Fisher and Seborg (1972) to test order reduction methods:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{5.1}$$

For Davison's method (3.8) the reduced order model is

$$\dot{\tilde{x}}_1 = -\tilde{x}_1 + u/3 \quad (5.2)$$

and for Marshall's method (3.14) the reduced order model is

$$\dot{\tilde{\tilde{x}}}_1 = -\tilde{\tilde{x}}_1 + u/4 \quad (5.3)$$

The Riccati equation (2.3) and Lyapunov equation (2.4) become scalar equations for this problem with solutions

$$L = 1, \quad K = 1/3.$$

The new model (4.3) is therefore given by

$$\begin{aligned} \dot{y}_1 &= -y_1 + u/3 \\ y_1(0) &= 4x_1(0)/3 + x_2(0)/3 \\ \hat{y}_2(t) &= u(t)/4, \\ \hat{x}_1(t) &= y_1(t) - \hat{y}_2(t)/3 \\ \hat{x}_2(t) &= -y_1(t) + 4\hat{y}_2(t)/3 \end{aligned} \quad (5.4)$$

The response of the three models to initial conditions  $(x_1(0), x_2(0)) = (1, 1)$  is illustrated in Figure 2.1. The new model is clearly superior here as (4.3d) provides the true projection of initial conditions  $x(0)$  onto the

- I = True solution  
II = Davison's model  
III = Marshall's model  
IV = New model

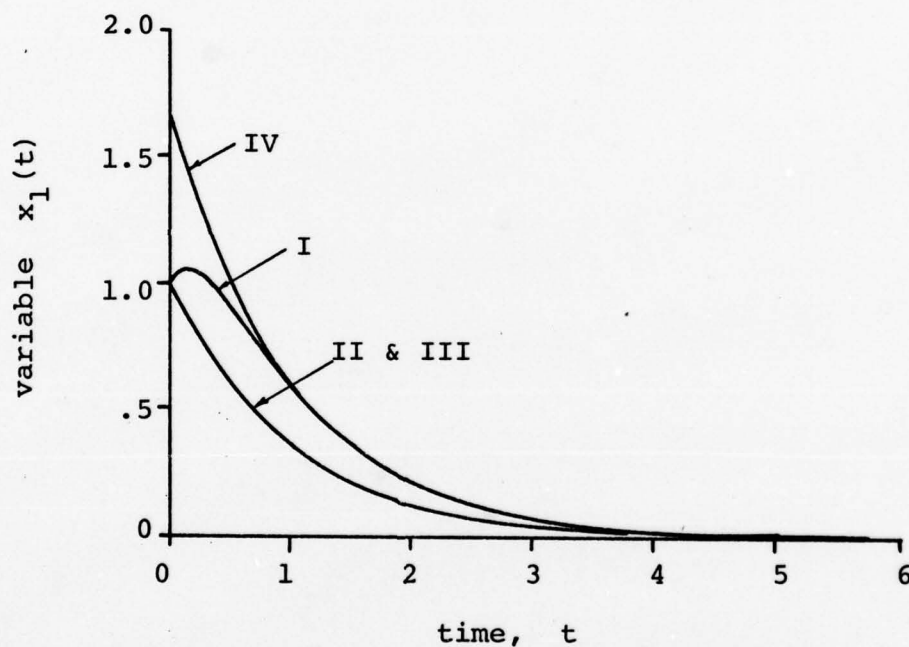


Figure 2.1. Response of reduced-order models to initial conditions for Example 2.5.

slow eigenspace (i.e., the  $n_1$  dimensional space spanned by the slow eigenvectors) whereas neither of the other models provide this feature. Since the LK transformation  $T$  is a time-scale-decoupling transformation  $\Gamma$  (cf. (I.1.7)) which exactly decomposes  $n$ -dimensional space into fast and slow eigenspaces, the new model correctly projects the slow mode  $y_1(t)$  back onto the original state  $x(t)$ .

The responses of the three models to unit step input  $u(t) = 1$ , ramp input  $u(t) = t$ , and sinusoidal input  $u(t) = \sin(\frac{\pi}{3}t)$  are illustrated in Figures 2.2 through 2.4. The boundary layer structure of the fast mode corresponding to eigenvalue  $-4$  can be clearly seen in Figures 2.1 and 2.2 as the difference between the true solution and the new model. In the first three cases the new model provides a more accurate approximation to the true solution than either Davison's or Marshall's models. For the case of sinusoidal input the new model and Marshall's model lead and lag the true response, respectively, and might be considered equally accurate. Note that in all cases since the new model approximates the fast mode as asymptotically fast, i.e., the time scale in (4.2b) shrinks to zero, the new model leads the true response. Since these models are only intended to represent the slow mode dynamics of the system, they can only be expected to be accurate for low frequency inputs, i.e., frequencies less than the order of magnitude of the time

- I = True solution  
II = Davison's model  
III = Marshall's model  
IV = New model

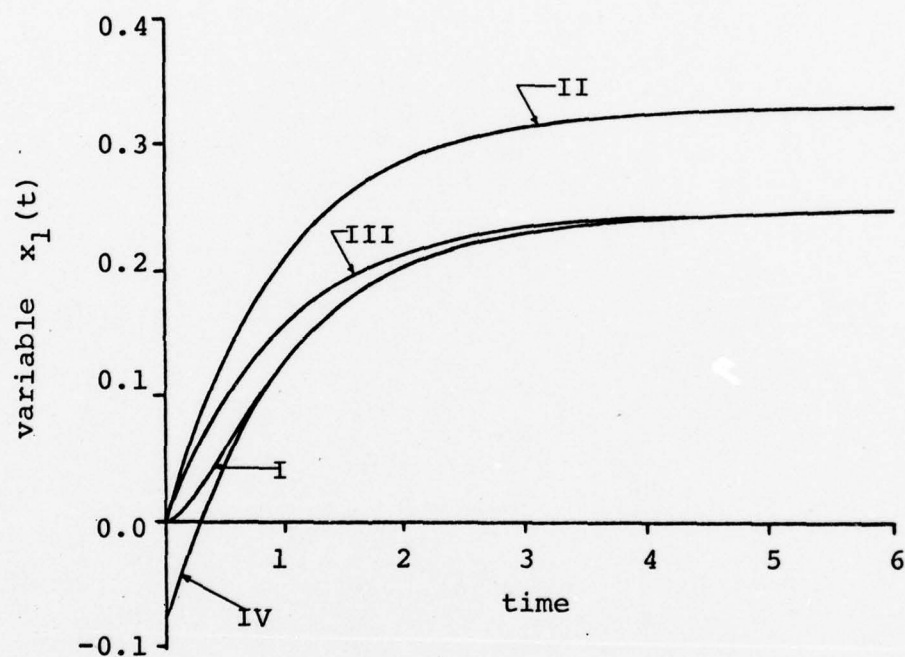


Figure 2.2. Response of reduced-order models to a step input for Example 2.5.

- I = True solution  
II = Davison's model  
III = Marshall's model  
IV = New model

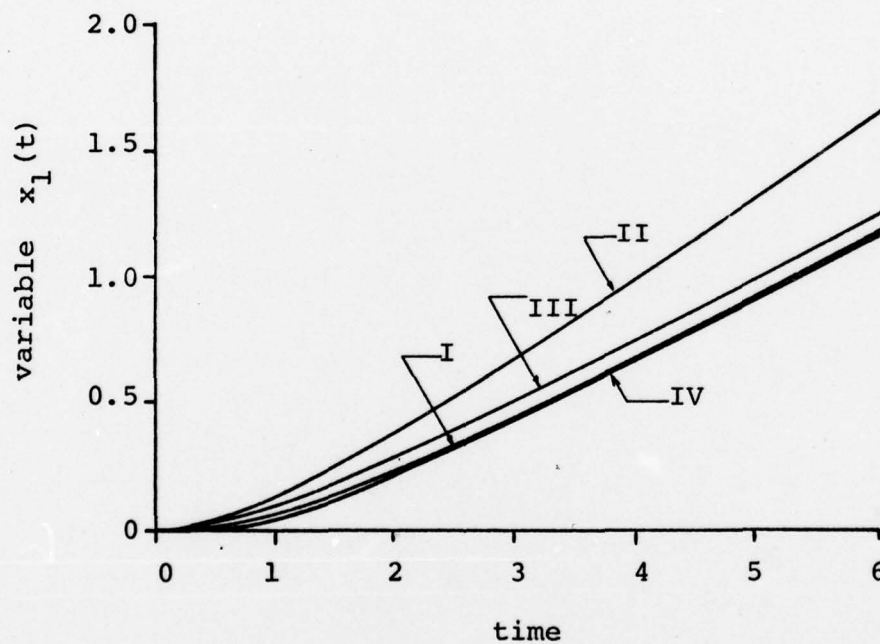


Figure 2.3. Response of reduced-order models to a ramp input for Example 2.5.

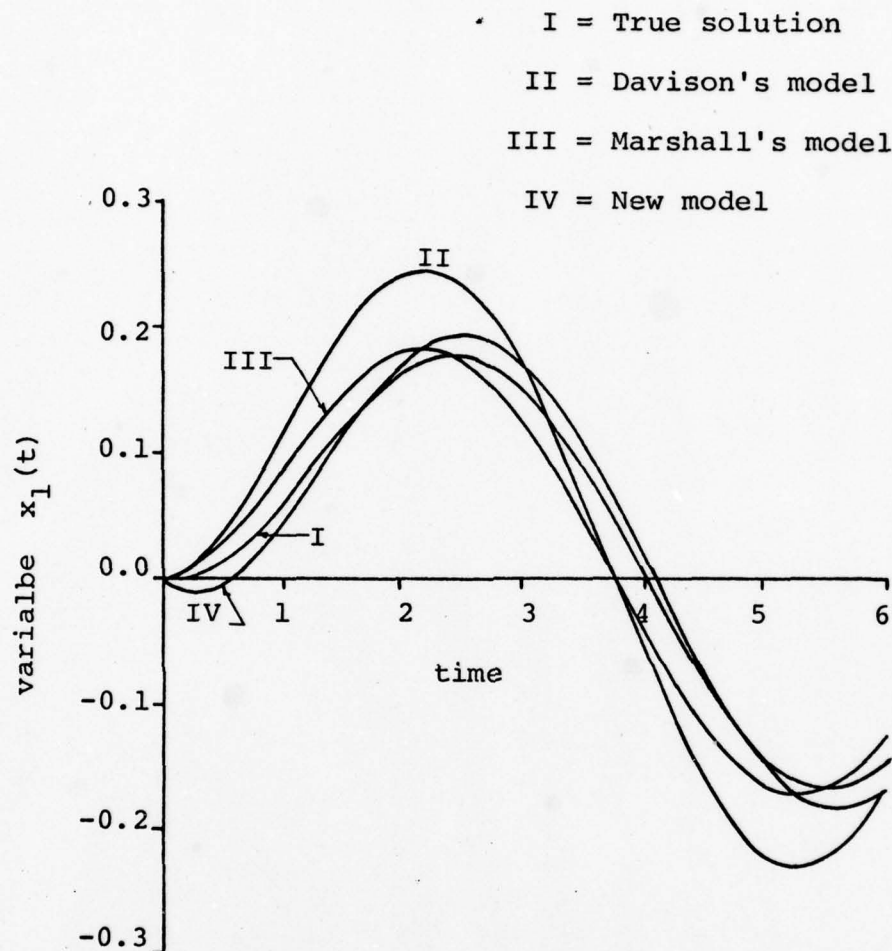


Figure 2.4. Response of reduced-order models to a sinusoidal input for Example 2.5.

constant  $1/|s_{n_1}|$ . The new order reduction method and Marshall's method will be compared again in Chapter 5 when they are employed to model the slow mode dynamics of an F8 aircraft.

### 2.6. Feedback Matrix Design via the LK Transformation

Often dynamic systems are modeled as linear control systems for the purpose of designing a feedback controller to achieve specific closed-loop eigenvalue locations. As will be shown, this task can be greatly simplified by use of a time scale decoupling transformation such as the LK transformation.

It is our purpose here to show that given any time-scale decoupling transformation  $\Gamma$ , such as  $T$ , an  $n$ -dimensional eigenvalue placement problem can be reduced to separate eigenvalue placement problems of dimensions  $n_1$  and  $n_2$ . Here the control matrix  $B$  and the output matrix  $C$  will be taken to be constant, and  $n_1$  and  $n_2$  could correspond to any weakly two-time-scale division of eigenvalues.

It is well known that if a constant linear control system is controllable, i.e.,

$$\text{rank}(B \begin{vmatrix} A \\ AB \\ \dots \\ A^{n-1}B \end{vmatrix}) = n, \quad (6.1)$$

then there exists at least one real  $m \times n$  dimensional

feedback matrix  $H$  such that the closed-loop eigenvalues given by

$$\lambda(A + BH) \quad (6.2)$$

can be placed arbitrarily as long as complex eigenvalues appear in conjugate pairs.

If the original system is controllable then the decoupled slow and fast systems (2.5) are also controllable. This can be demonstrated as follows. Let the block-decoupling transformation  $T$  premultiply the matrix (6.1) to yield

$$\begin{aligned} & [TB_1' (TAT^{-1}) TB_1' \dots (TAT^{-1}) TB_1' \dots (TAT^{-1}) TB_2' \dots (TAT^{-1}) TB_2'] \\ &= \begin{bmatrix} \tilde{B}_1' & \begin{bmatrix} \tilde{A}_{11} & 0 \end{bmatrix} & \begin{bmatrix} \tilde{B}_1' \end{bmatrix} & \vdots & \begin{bmatrix} \tilde{A}_{11}^{n-1} & 0 \end{bmatrix} & \begin{bmatrix} \tilde{B}_1' \end{bmatrix} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_2' & \begin{bmatrix} 0 & \tilde{A}_{22} \end{bmatrix} & \begin{bmatrix} \tilde{B}_2' \end{bmatrix} & \vdots & \begin{bmatrix} 0 & \tilde{A}_{22}^{n-1} \end{bmatrix} & \begin{bmatrix} \tilde{B}_2' \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{B}_1' & \tilde{A}_{11} & \tilde{B}_1' & \vdots & \tilde{A}_{11}^{n-1} & \tilde{B}_1' \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{B}_2' & \tilde{A}_{22} & \tilde{B}_2' & \vdots & \tilde{A}_{22}^{n-1} & \tilde{B}_2' \end{bmatrix}. \end{aligned}$$

If the  $n \times nm$  matrix (6.1) is of full rank, then the matrices

$$[\hat{B}_1 | \hat{A}_{11} \hat{B}_1 | \dots | \hat{A}_{11}^{n-1} \hat{B}_1], \quad (6.3a)$$

$$[\hat{B}_2 | \hat{A}_{22} \hat{B}_2 | \dots | \hat{A}_{22}^{n-1} \hat{B}_2] \quad (6.3b)$$

are each of full rank, and controllability of the pair  $(A, B)$  implies controllability of  $(\hat{A}_{11}, \hat{B}_1)$  and  $(\hat{A}_{22}, \hat{B}_2)$ . However, the converse is not true since for arbitrary  $\hat{A}_{11}$ ,  $\hat{B}_1$ ,  $\hat{A}_{22}$ ,  $\hat{B}_2$  full rank matrices (6.3a) and (6.3b) could have linearly dependent rows. Conditions under which controllability of the fast and slow subsystems implies controllability of the original system are described by Chow (1977) and Sannuti (1978). (If the pair  $(A^t, C^t)$  is controllable where  $C$  is the system output matrix in (4.6) then the control system (4.6) is observable. If the original control system is observable, then it can be shown by duality that the decoupled control systems (4.1) are also observable.) For the remainder of this section, we will assume that the original system (4.6) is controllable.

The design of a feedback matrix  $H$  to achieve a specific closed-loop eigenvalue location is very straightforward. Suppose that it is desired to relocate the  $n_1$  slow eigenvalues  $S$  to  $n_1$  new eigenvalue locations  $S'$ . If one can find an  $m \times n_1$  feedback matrix  $\hat{H}_1$  which satisfies

$$\lambda(\hat{A}_{11} + \hat{B}_1 \hat{H}_1) = S' \quad (6.4)$$

then the system (4.1) with feedback

$$u = \hat{H}_1 y_1$$

implies that

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} + \tilde{B}_1 \hat{H}_1 & 0 \\ \tilde{B}_2 \hat{H}_1 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \quad (6.5)$$

Thus the fast eigenvalues  $F$  are unchanged and the corresponding feedback matrix  $H$  for the original system (4.6) can be found as

$$u = Hx \text{ for } H = [\hat{H}_1 \quad 0]^T \quad (6.6)$$

since  $y = Tx$ .

If in addition to relocating the slow mode eigenvalues  $S$  it is also necessary to relocate the open-loop fast eigenvalues  $F$  to closed-loop values  $F'$ , then this can be done as follows. After solving for feedback matrix  $\hat{H}_1$  in (6.4), find the unique (assuming  $S'$  and  $F$  have no common eigenvalues, Gantmacher 1959, p. 225)  $n_2 \times n_1$  matrix  $P$  satisfying the Lyapunov equation

$$P(\hat{A}_{11} + \hat{B}_1 \hat{H}_1) - \hat{A}_{22}P + \hat{B}_2 \hat{H}_1 = 0. \quad (6.7)$$

The block-triangular form (6.5) can then be transformed to a new block-diagonal form by introducing new fast variables

$$\hat{y}_2 = Py_1 + y_2. \quad (6.8)$$

Now relocate the fast eigenvalues  $F$  to new values  $F'$  by finding an  $m \times n_2$  feedback matrix  $\hat{H}_2$  satisfying

$$\lambda(\hat{A}_{22} + (P\hat{B}_1 + \hat{B}_2)\hat{H}_2) = F'. \quad (6.9)$$

The appropriate gain matrix for the original state variable  $x$  is then given by

$$u = [(\hat{H}_1 - \hat{H}_2P) \quad \hat{H}_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Hx \quad (6.10)$$

where  $H = [(\hat{H}_1 - \hat{H}_2P) \quad \hat{H}_2]T$ . Thus the block-decoupling transformation  $T$  can be used to exactly relocate both slow and fast eigenvalues separately via state feedback. Approximate methods of locating slow eigenvalues (Georgakis and Bauer, 1978), or separately locating slow and fast eigenvalues (Chow and Kokotovic, 1976) based on the singularly perturbed system (3.16) have also been proposed.

Many variations of this approach are possible to facilitate the feedback matrix design task. For example, by repeated application of the block-decoupling transformation, either of the decoupled subsystems in (4.1) could be further transformed into block-diagonal form and a complete eigenvalue relocation could be achieved in three consecutive design steps analogous to the above two design steps (6.4) and (6.9). In many practical applications (e.g., Skira and DeHoff, 1977) the fast modes are all stable and well damped as defined in (3.1) and (3.2) so only the slow mode eigenvalues need be relocated.

The practicality of this procedure depends directly upon being able to calculate the appropriate  $L$  and  $K$  matrices. As will be shown in Chapter 4, accurate approximations to the matrices may be readily computed.

## CHAPTER 3

### EXISTENCE AND UNIQUENESS OF THE L AND K TIME-SCALE DECOUPLING MATRICES

In this chapter the general solution to the non-symmetric algebraic Riccati equation (ARE), (II.2.3) is described. As shown in Chapter 2, the task of finding an  $(n - n_1) \times n_1$  L matrix such that transformation

$$\begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

reduces a partitioned linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

to block-triangular form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ 0 & A_{22} + LA_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix}$$

is equivalent to solving the ARE. Past work on the existence of solutions to ARE is reviewed, and general conditions for existence of  $L$  which must be satisfied by matrix  $A$  and by the integer  $n_1$  (where  $1 \leq n_1 < n$ ) are presented in Theorem 3.1. It is shown that  $L$  can be represented in terms of eigenvectors and generalized eigenvectors of  $A$ , and that there is a simple relationship between the eigenvalues of  $\tilde{A}_{11} = A_{11} - A_{12}L$  and these eigenvectors.

In order to describe the solution  $L$  to ARE we first need to review the eigenstructure of square matrices. After the solution to ARE is presented, it is shown that the matrix  $K$  satisfying the Lyapunov equation (II.2.4) exists whenever  $L$  exists. In Section 3.3 conditions for  $L$  and  $K$  to be real are described. It is then shown in Theorem 3.2 that for a given (weakly) two-time-scale system, the  $L$  matrix is unique. From these results it follows that time-scale-decoupling transformation  $T$  (II.2.2) exists, is real and unique for any given (weakly) two-time-scale system.

### 3.1. A Review of Matrix Eigenstructure

At this point it is not assumed that  $A$  has any special properties such as the weakly (or strongly) two-time-scale property, but only that it is a real square matrix of dimension  $n \times n$  with  $p$  distinct eigenvalues  $\lambda_1, \dots, \lambda_p$ .

For all matrices  $A$  there exists a nonsingular modal matrix,  $M$ , which transforms  $A$  to Jordan (canonical) form

$$M^{-1}AM \equiv \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{bmatrix} \equiv J \quad (1.1)$$

(note  $J_i$  is defined in (II.3.3) and the  $J_i$  are  $m_i \times m_i$  Jordan blocks (Noble and Daniel, 1977, p. 368). In general,  $p \leq k$ , and the Jordan block  $J_i$  corresponding to the eigenvalue  $\lambda_j$  will be indicated as  $J_{i,\lambda_j}$ . A Jordan block is a square matrix of the form

$$J_{i,\lambda_j} = \begin{bmatrix} \lambda_j & 1 & & & \\ & \lambda_j & 1 & & \\ & & \lambda_j & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_j \end{bmatrix} \quad (1.2)$$

(If  $m_i = 1$ , the one's do not appear.) The number  $k$  of Jordan blocks, the numbers  $m_1, \dots, m_k$  and the values  $\lambda_1, \dots, \lambda_p$  are uniquely determined by  $A$ . However,

the ordering of Jordan blocks  $J_i$  within  $J$  is arbitrary.

If  $\lambda_j$  is an eigenvalue with multiplicity  $\ell_j$ , let  $J_{\lambda_j}$  represent the  $\ell_j \times \ell_j$  matrix formed from the Jordan blocks in  $J$  with eigenvalue  $\lambda_j$ , and let  $M_{\lambda_j}$  represent the  $n \times \ell_j$  matrix formed from the  $\ell_j$  columns of  $M$  corresponding to  $J_{\lambda_j}$ , so that

$$AM_{\lambda_j} = M_{\lambda_j} J_{\lambda_j}. \quad (1.3)$$

The  $\ell_j$ -dimensional eigenspace of  $C^n$  spanned by the columns of  $M_{\lambda_j}$  is uniquely determined by  $A$ .

Taking a complex conjugate (indicated by an overbar) of (1.3),

$$A\overline{M}_{\lambda_j} = \overline{M}_{\lambda_j} \overline{J}_{\lambda_j} \quad (1.4)$$

implies that if  $J_{i,\lambda_j}$  is in  $J$ , then  $\overline{J}_{i,\lambda_j}$  is also in  $J$ . Also, without loss of generality,  $\overline{M}_{\lambda_j}$  may be represented as  $\overline{M}_{\lambda_j}$ , assuming the ordering of blocks within  $J_{\lambda_j}$  and  $\overline{J}_{\lambda_j}$  is the same.

Corresponding to each Jordan block  $J_{i,\lambda_j}$  there is a (right) eigenvector  $v_{i,\lambda_j}$  which is a column of  $M$

satisfying

$$(A - \lambda_j I)v_{i,\lambda_j} = 0. \quad (1.5)$$

If  $\lambda_j$  is a distinct eigenvalue, then  $v_{i,\lambda_j}$  is unique up to a multiplicative complex scalar constant. This non-uniqueness can be removed by normalization.

If  $\lambda_j$  has multiplicity greater than one, then any linear combination of the eigenvectors of  $\lambda_j$  is also an eigenvector. If  $m_i > 1$ , then corresponding to the Jordan block  $J_{i,\lambda_j}$  there are generalized (right) eigenvectors  $v_{i,\lambda_j}^r$  of rank  $r$  for  $r = 2, \dots, m_i$  which satisfy

$$(A - \lambda_j I)^r v_{i,\lambda_j}^r = 0 \quad (1.6)$$

$$(A - \lambda_j I)^{r-1} v_{i,\lambda_j}^r \neq 0.$$

These generalized eigenvectors are also columns of  $M$  corresponding to  $J_{i,\lambda_j}$ . Any linear combination of generalized eigenvectors of lower rank may be added to a generalized eigenvector and (1.6) will still be satisfied. One straightforward procedure for computing the generalized eigenvectors is described by Chen (1970).

### 3.2. The General Solution to the Nonsymmetric Algebraic Riccati Equation

Given matrix  $A$  and an integer  $n_1$  satisfying  $1 \leq n_1 < n$ , we can determine the transformed matrix  $\tilde{A}$  by

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \equiv \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix}, \quad (2.1)$$

and  $\tilde{A}_{21} = 0$  is equivalent to the ARE

$$LA_{11} - A_{22}L - LA_{12}L + A_{21} = 0. \quad (2.2)$$

In his previous study of ARE, Kokotovic (1975) used a fixed point theorem to show that if the norm inequality

$$\|A_{22}^{-1}\| \leq \frac{1}{3}(\|A_{11} - A_{12}A_{22}^{-1}A_{21}\| + \|A_{12}\|\|A_{22}^{-1}A_{21}\|)^{-1}$$

is satisfied (assuming  $A_{22}^{-1}$  exists) then there exists a real solution to ARE which is the unique limit of the sequence of matrices

$$L_0 = A_{22}^{-1}A_{21}, \quad (2.3)$$

$$L_{i+1} = A_{22}^{-1}(A_{21} + L_i A_{11} - L_i A_{12} L_i), \quad i = 0, 1, 2, \dots$$

In another study Narasimhamurthi and Wu (1977) proved that if all the eigenvalues of  $A$  are distinct, then there exists a complex matrix satisfying ARE. Also, Veljko (1977) presented the solution to the ARE for the special case where each multiple eigenvalue has only a single eigenvector. Both of these works described the relationship between the matrix  $L$  satisfying ARE and the eigenvectors of  $A$ . These results are included in Theorem 3.1 which describes general necessary and sufficient conditions for the existence of a matrix  $L$  satisfying ARE. The key results of Sections 3.2-3.4 were presented in Anderson (1978).

Potter (1966), Martensson (1971) and Kucera (1972) have also described the eigenstructure of  $L$  for the special case of the symmetric algebraic Riccati equation from the optimal linear regulator problem. For the symmetric Riccati equation  $n_1 = n_2$ ,  $A_{11} = A_{22}^T$  and  $A_{12}$ ,  $A_{21}$  are symmetric and positive semi-definite. In the large, our analysis parallels theirs.

Theorem 3.1. Matrix  $L$  satisfies the ARE if and only if  
(1) some subset of the numbers  $\{m_i\}$  can be chosen so that

$$m_i + \dots + m_j = n_1. \quad (2.4)$$

Then

$$\mathcal{Y}_1 \equiv \begin{bmatrix} J_i & & \\ & \ddots & \\ & & J_j \end{bmatrix} \quad (2.5)$$

is an  $n_1 \times n_1$  matrix and

$$\mathcal{Y}_2 \equiv \begin{bmatrix} J'_i & & \\ & \ddots & \\ & & J'_j \end{bmatrix} \quad (2.6)$$

is an  $n_2 \times n_2$  matrix where  $i', \dots, j'$  are the indices between 1 and  $k$  not used in (2.5). Also the matrices  $M$  and  $Q = M^{-1}$  are partitioned compatibly with  $\mathcal{Y}_1$  and  $\mathcal{Y}_2$  so that

$$A = M\mathcal{Y}Q = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \mathcal{Y}_1 & 0 \\ 0 & \mathcal{Y}_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}. \quad (2.7)$$

(2)  $M_{11}$  is of full rank.

(3)  $LM_{11} = -M_{21}$ . (2.8)

Proof. Assume first that  $L$  satisfies ARE (2.2). There exists an  $n_1 \times n_1$  matrix  $G_1$  in Jordan form and a non-singular matrix  $X$  such that

$$A_{11} - A_{12}L = XG_1X^{-1} \quad (2.9)$$

and ARE can be rewritten as

$$A_{21} - A_{22}L = -L(A_{11} - A_{12}L) = -LXG_1X^{-1}. \quad (2.10)$$

Postmultiply (2.9) and (2.10) by  $X$  and let  $Y = -LX$  so that

$$A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} G_1.$$

Thus  $G_1$  is an  $n_1 \times n_1$  block of the Jordan canonical form of  $A$ ,  $X$  is of full rank, and the  $L$  matrix satisfies the linear equation  $LX = -Y$  formed from the appropriately partitioned columns of a modal matrix corresponding to  $G_1$ . As described earlier, the modal matrix is determined only up to the choice of basis vectors of the eigenspaces, so without loss of generality we can choose  $M_{11}$ ,  $M_{21}$  as  $X$ ,  $Y$ . This completes the first half of the proof.

Now assume (1), (2) and (3) above are satisfied. We need to show that the lower left block  $\tilde{A}_{21}$  of (2.1) is zero. Since  $QM = I$ , it follows that

$$Q_{21}M_{11} + Q_{22}M_{21} = 0$$

so

$$Q_{21} = -Q_{22}M_{21}M_{11}^{-1},$$

and

$$Q_{22}L = Q_{21}. \quad (2.11)$$

Since the  $n_2$  rows of  $[Q_{21} \ Q_{22}]$  are of full rank, it follows that  $Q_{22}[L \ I]$  and  $Q_{22}$  are both of full rank. Write  $A$  as (2.7) and complete the product (2.1) so that

$$\begin{aligned} & \begin{bmatrix} I & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_{22}^{-1}Q_{21} & I \end{bmatrix} \\ &= \begin{bmatrix} M_{11}J_1M_{11}^{-1} & M_{11}J_1Q_{12} + M_{12}J_2Q_{22} \\ 0 & Q_{22}^{-1}J_2Q_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad (2.12) \end{aligned}$$

since

$$M_{11}^{-1} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21} \quad (2.13a)$$

and

$$Q_{22}^{-1} = M_{22} - M_{21}M_{11}^{-1}M_{12}. \quad (2.13b)$$

This completes the proof.

This theorem provides the key result which will be used later when computing the  $L$  matrix; namely, if  $L$  satisfies (2.8) then

$$\hat{A}_{11}M_{11} = M_{11}\hat{J}_1, \quad (2.14a)$$

and

$$\hat{A}_{22}Q_{22}^{-1} = Q_{22}^{-1}\hat{J}_2 \quad (2.14b)$$

so the eigenvalues and eigenvectors of  $\hat{A}_{11}$  and  $\hat{A}_{22}$  are determined via  $\hat{J}_1$ ,  $M_{11}$  and  $\hat{J}_2$ ,  $Q_{22}^{-1}$ , respectively. The case of generalized eigenvectors causes no difficulty since any Jordan block appearing in  $G_1$  also appears in  $\hat{J}_1$  and conversely.

Theorem 3.1 shows that under the very special condition that either (1) or (2) above is not satisfied, solutions to ARE may not exist. However, the (weakly) two-time-scale property guarantees that (1) can be satisfied, and if one is

allowed to reindex the state variables  $x$ , then (2) can also be satisfied.

Recall that if  $L$  satisfies ARE and there exists a matrix  $K$  satisfying

$$K\hat{A}_{22} - \hat{A}_{11}K + A_{12} = 0 \quad (2.15)$$

then  $\tau A \tau^{-1} = \hat{A}$  is block-diagonal. Such a  $K$  satisfying (2.15) can be explicitly represented in terms of the modal matrix. Since  $MQ = I$ ,

$$M_{11}Q_{12} + M_{12}Q_{22} = 0,$$

expressing  $\hat{A}_{11}$ ,  $\hat{A}_{22}$  and  $A_{12}$  in spectral form, it follows that (2.15) is satisfied by

$$K = M_{11}Q_{12} = -M_{12}Q_{22} \quad (2.16)$$

i.e.,

$$\begin{aligned} & (-M_{12}Q_{22})(Q_{22}^{-1}J_2Q_{22}) - (M_{11}J_1M_{11}^{-1})(M_{11}Q_{12}) \\ & + (M_{11}J_1Q_{12} + M_{12}J_2Q_{22}) = 0. \end{aligned}$$

Thus  $K$  exists whenever  $L$  exists.

We then have the alternative expressions

$$L = Q_{22}^{-1} Q_{21} = -M_{21} M_{11}^{-1}. \quad (2.17)$$

Applying identities (2.14) we can express the LK transformation  $T$  and its inverse in terms of partitions of the modal matrix, viz.,

$$T = \begin{bmatrix} I + KL & K \\ L & I \end{bmatrix} = \begin{bmatrix} M_{11} Q_{11} & M_{11} Q_{12} \\ Q_{22}^{-1} Q_{21} & I \end{bmatrix}, \quad (2.18)$$

and

$$T^{-1} = \begin{bmatrix} I & -K \\ -L & I + LK \end{bmatrix} = \begin{bmatrix} I & M_{12} Q_{22} \\ M_{21} M_{11}^{-1} & M_{22} Q_{22} \end{bmatrix}. \quad (2.19)$$

### 3.3. Conditions for Matrices L and K to be Real

One of the primary limitations to the usefulness of the modal matrix  $M$  and its inverse  $Q$  is that they are complex whenever the system has complex eigenvalues. As shown below, under certain restrictions, the  $L$  matrix satisfying ARE will be real.

In order to facilitate the presentation, we will assume that the Jordan blocks within the Jordan form  $\tilde{J}_1$  of (2.5) are ordered so that

$$\tilde{J}_1 = \begin{bmatrix} J_0 & & \\ & J_+ & \\ & & J_- \end{bmatrix}$$

where  $J_0$ ,  $J_+$  and  $J_-$  correspond respectively to Jordan blocks in  $\tilde{J}_1$  with  $\text{Im}(\lambda_i) = 0$ ,  $\text{Im}(\lambda_i) > 0$ , and  $\text{Im}(\lambda_i) < 0$ , and the dimensions of  $J_0$ ,  $J_+$  and  $J_-$  are  $m_0 \times m_0$ ,  $m_+ \times m_+$  and  $m_- \times m_-$ , with  $m_0 + m_+ + m_- = n_1$ . Also, let the  $m_0$  columns of  $M$  corresponding to  $J_0$  be represented as  $M_0$  and likewise for  $M_+$  and  $M_-$  so that

$$A \begin{bmatrix} M_0 & M_+ & M_- \end{bmatrix} = \begin{bmatrix} M_0 & M_+ & M_- \end{bmatrix} \begin{bmatrix} J_0 & & \\ & J_+ & \\ & & J_- \end{bmatrix}.$$

Then if

$$J_- = \bar{J}_+, \quad (3.1)$$

the  $L$  and  $K$  matrices satisfying ARE and (2.15) are real.

This can be shown as follows. Let  $M_-$  be represented as  $\bar{M}_+$ . By Theorem 3.1,  $L$  satisfies  $LM_{11} = -M_{21}$ . Postmultiply  $[M_0 \ M_+ \ \bar{M}_+]$  by the nonsingular matrix

$$\tilde{C}_1 \equiv \frac{1}{2} \begin{bmatrix} 2I_{m_0} & 0 & \\ 0 & I_{m_+} & -iI_{m_+} \\ 0 & I_{m_+} & iI_{m_+} \end{bmatrix},$$

where for clarity the dimensions of the identity matrices are explicitly noted, to obtain the real matrix

$$\begin{bmatrix} \hat{M}_{11} \\ \hat{M}_{21} \end{bmatrix} \equiv \begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \tilde{C}_1.$$

Thus  $L$  satisfies

$$LM_{11}\tilde{C}_1 = -M_{21}\tilde{C}_1$$

or  $L\hat{M}_{11} = -\hat{M}_{21}$  and  $L$  is therefore real.

If the Jordan blocks of  $\mathcal{Y}_1$  appear in complex conjugate pairs, cf. (3.1), then so do those of  $\mathcal{Y}_2$ . By the above argument, there also exists a nonsingular complex matrix  $\tilde{C}_2$  of dimension  $n_2 \times n_2$  such that

$$\begin{bmatrix} \tilde{M}_{12} \\ \tilde{M}_{22} \end{bmatrix} = \begin{bmatrix} M_{12} \\ M_{22} \end{bmatrix} \tilde{C}_2 \quad (3.2)$$

is real. Equations (2.16) and (3.2) imply that

$$\begin{aligned} K &= -M_{12}Q_{22} = M_{12}(M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1} \\ &= -M_{12}\tilde{C}_2\tilde{C}_2^{-1}(M_{22} - M_{21}\tilde{C}_1(M_{11}\tilde{C}_1)^{-1}M_{12})^{-1} \\ &= -\tilde{M}_{12}(\tilde{M}_{22} - \tilde{M}_{21}\tilde{M}_{11}^{-1}\tilde{M}_{12})^{-1} \end{aligned}$$

so  $K$  is also real under conditions (3.1).

From these results it follows that if the LK transformation  $T$  decouples the time scales of a (weakly) two-time scale system, then  $T$  is real.

#### 3.4. Uniqueness of the $L$ and $K$ Matrices

As shown by Narasimhamurthi and Wu (1977) the ARE will have a unique solution only under the very restrictive condition that the quadratic term is zero and the equation becomes linear. In fact, we will show below that there are in general many solutions of ARE. However, the  $L$  and  $K$  matrices in  $T$  which decouple a given (weakly) two-time-scale

system so that slow and fast modes are separated are unique. This is proven in the following theorem.

Theorem 3.2. For a given (weakly) two-time-scale system (with  $n_1$  specified), there exists at most one decoupling matrix  $L$  satisfying both ARE and

$$\lambda(\hat{A}_{11}) = S, \quad \lambda(\hat{A}_{22}) = F. \quad (4.1)$$

Proof. Assume that  $L$  and  $L'$  both satisfy ARE and (4.1). Applying Theorem 3.1 there exist full rank matrices

$$\begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X' \\ Y' \end{bmatrix} \quad (4.2)$$

such that  $LX = -Y$  and  $L'X' = -Y'$ . Moreover, the columns of (4.2) are (possibly generalized) eigenvectors of  $A$  corresponding to eigenvalues  $S$ . Since there is a unique  $n_1$  dimensional eigenspace spanned by these (generalized) eigenvectors, there exists a nonsingular matrix  $\mathcal{Z}$  such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix} \mathcal{Z}.$$

Postmultiply  $L'X' = -Y'$  by  $\tilde{C}$  to obtain  $L'X = -Y$ . From Theorem 1,  $X$  is of full rank, so  $L = L' = -YX^{-1}$ , completing the proof.

The uniqueness of the  $K$  matrix is insured by the two-time-scale property. As shown by Gantmacher (1959), the Lyapunov equation (2.15) has a unique solution provided  $\tilde{A}_{11}$  and  $\tilde{A}_{22}$  have no common eigenvalues.

Since  $L$  and  $K$  are uniquely determined, the decoupling transformation  $T$  is thereby unique.

Although the transformation  $T$  is unique, there are generally many real solutions to ARE. Thus any algorithm to obtain  $L$  should avoid converging to these other solutions. In a special case, the number of distinct solutions can be enumerated.

Corollary 3.3. If  $A$  has (real) distinct eigenvalues and each set of  $n_1$  columns taken from the  $n$  columns  $[M_{11} \ M_{12}]$  is of full rank, then there are  $\binom{n}{n_1}$  distinct (real) solutions of ARE corresponding to the number of distinct ways of partitioning the  $n$  eigenvalues into sets of  $n_1$  and  $n_2$  members.

Proof. By Theorem 3.1 and 3.2 there exists one solution for each of the  $\binom{n}{n_1}$  partitions of eigenvalues. Then  $\lambda(A_{11} - A_{12}L) \neq \lambda(A_{11} - A_{12}L')$  implies  $L \neq L'$ , so these solutions are distinct.

## CHAPTER 4

### COMPUTING THE LK TRANSFORMATION

If efficient eigenanalysis programs are available, a computational procedure for obtaining accurate numerical approximations to the L and K matrices is quite straightforward. One such program, EISPACK, is available from the Argonne National Laboratory and is described by Smith et al. (1976). This program consists of many modular subroutines which can handle a variety of eigenanalysis tasks. It is well documented and easily implemented on either batch or timesharing computers. Typical execution times for finding all eigenvalues and eigenvectors of matrices of order 40 and 80 are .66 and 4.6 seconds for an IBM 370/195 computer.

#### 4.1. Computing the L Matrix

Since the L matrix can be expressed as either

$$LM_{11} = -M_{21} \quad (1.1)$$

or

$$Q_{22}L = Q_{21} \quad (1.2)$$

not all eigenvectors of  $A$  need be obtained to solve for  $L$ . Two procedures could be used, one for the case  $n_1 \leq n_2$  and another for  $n_1 > n_2$ . For  $n_1 \leq n_2$ , obtain  $n_1$  (right) eigenvectors

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$

corresponding to the slow eigenvalues  $S$  and solve the set of linear equations (1.1) for the  $L$  matrix. Such equations can be efficiently solved by a Gaussian elimination scheme such as the LU decomposition, cf. Stewart (1973). In the event that  $S$  contains one or more complex eigenvalue pairs  $\lambda_i, \bar{\lambda}_i$  with corresponding complex eigenvectors  $v_{\lambda_i}, v_{\bar{\lambda}_i}$  replace  $v_{\lambda_i}$  and  $v_{\bar{\lambda}_i}$  with  $\text{Re}(v_{\lambda_i})$  and  $\text{Im}(v_{\lambda_i})$ . Then assuming the states are ordered so that the resulting  $M_{11}$  is of full rank, solve (1.1). A method of approximately minimizing the condition number of  $M_{11}$ , avoiding a singular matrix  $M_{11}$  and obtaining an  $L$  matrix of small norm is presented in Section 4.3.

For  $n_1 > n_2$ , an alternate procedure could be followed yielding fewer computations. That is, find the  $n_2$  left eigenvectors  $[Q_{21} \ Q_{22}]$  of  $A$  and, after eliminating imaginary parts, solve (1.2) for  $L$ . Here the term left eigenvector is well defined by the notation and equals the

corresponding eigenvector of  $A^T$ . This terminology differs somewhat from that of some texts which refer to the rows of  $\bar{Q}$  as the left eigenvectors of  $A$ .

Let  $L_0$  be the matrix obtained by approximately solving either (1.1) or (1.2). The accuracy of  $L_0$  can be checked by substituting it back into the nonsymmetric algebraic Riccati equation and evaluating the residual error matrix

$$R_0 = L_0 A_{11} - A_{22} L_0 - L_0 A_{12} L_0 + A_{21}. \quad (1.3)$$

If  $R_0$  is judged to be not sufficiently small, i.e., if  $\|R_0\| > \kappa$  for some small tolerance  $\kappa > 0$  and some computable norm  $\|\cdot\|$ , then the accuracy of  $L_0$  can be improved by an iterative procedure.

An efficient method of improving the initial approximation  $L_0$  to the decoupling matrix  $L$  can be obtained by using a successive approximations scheme on ARE with  $L_0$  as the first iterate. Using the fact that  $\lambda(\tilde{A}_{22}) = F$  we can write ARE as

$$L = \tilde{A}_{22}^{-1} (L A_{11} + A_{21}) = (A_{22} + L A_{12})^{-1} (L A_{11} + A_{21}).$$

If  $L_0$  is a reasonable approximation to  $L$ , then the matrix  $(A_{22} + L_0 A_{12})$  has large eigenvalues, is nonsingular, and

the preceding equation suggests the linear iteration.

$$L_{i+1} = (A_{22} + L_i A_{22})^{-1} (L_i A_{11} + A_{21}).$$

This iterative correction takes specific advantage of the two-time-scale property of the system and can be implemented as

Algorithm 4.1.

- 1) Obtain the initial approximation  $L_0$  from either (1.1) or (1.2) and set  $i = 0$ .
- 2) Evaluate

$$R_i = L_i A_{11} - A_{22} L_i - L_i A_{12} L_i + A_{21}$$

and stop if  $\|R_i\| \leq \kappa$ .

- 3) Solve  $(A_{22} + L_i A_{12}) D_i = R_i$  for  $D_i$  and let  $L_{i+1} = L_i + D_i$ .
- 4) Let  $i = i + 1$  and go to 2.

It can be shown that for  $L_0$  close to  $L$ , the proposed iteration converges with approximate rate of convergence  $\mu$ . This follows since

$$\begin{aligned} R_{i+1} &= (L_i + D_i) A_{11} - A_{22} (L_i + D_i) \\ &\quad - (L_i + D_i) A_{12} (L_i + D_i) + A_{21} \end{aligned}$$

or

$$R_{i+1} = R_i + D_i(A_{11} - A_{12}L_i) - (A_{22} + L_iA_{12})D_i - D_iA_{12}D_i.$$

Taking the correction matrix  $D_i$  as in step 3,

$$\begin{aligned} R_{i+1} &= (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_i - A_{12}D_i) \\ &= (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_{i+1}). \end{aligned} \quad (1.4)$$

The largest eigenvalues in absolute value of  $(A_{22} + LA_{12})^{-1}$  and  $(A_{11} - A_{12}L)$  are  $\frac{1}{f_1}$  and  $s_{n_1}$ , so that the approximate rate of decrease in norm of the residual error matrix in Algorithm 4.1 is

$$\frac{\|R_{i+1}\|}{\|R_i\|} \sim \frac{|s_{n_1}|}{|f_1|} = \mu \ll 1. \quad (1.5)$$

This approximate rate of convergence can be further obtained as follows. Let  $\|\cdot\|$  be some consistent matrix norm, and let  $L_i$  differ from  $L$  by order  $\varepsilon$  for some  $0 \leq \varepsilon \ll 1$ , i.e.,

$$E_i \equiv L_i - L = O(\varepsilon).$$

Then equation (1.4) can be written as

$$R_{i+1} = (\hat{A}_{22} + E_i A_{12})^{-1} R_i (\hat{A}_{11} - A_{12} E_i). \quad (1.6)$$

Thus

$$\begin{aligned} R_{i+1} &= (\hat{A}_{22}^{-1} + 0(\epsilon)) R_i (\hat{A}_{11} + 0(\epsilon)) \\ &= \hat{A}_{22}^{-1} R_i \hat{A}_{11} + 0(\epsilon). \end{aligned} \quad (1.7)$$

Expressing  $\hat{A}_{11}$  and  $\hat{A}_{22}^{-1}$  in spectral form

$$\hat{A}_{11} = M_{11} J_1 M_{11}^{-1}, \quad \hat{A}_{22}^{-1} = Q_{22}^{-1} J_2^{-1} Q_{22}$$

where  $J_1 = \text{diag}(s_1, \dots, s_{n_1})$ ,  $J_2 = \text{diag}(f_1, \dots, f_{n_2})$ ,  
and pre- and post-multiplying (1.7) by  $Q_{22}$  and  $M_{11}$ , we  
obtain

$$Q_{22} R_{i+1} M_{11} = J_2^{-1} Q_{22} R_i M_{11} J_1 + 0(\epsilon). \quad (1.8)$$

If the  $A$  matrix is nondefective then

$$\|J_2^{-1}\| = \frac{1}{|f_1|} \quad \text{and} \quad \|J_1\| = |s_1|$$

for any of the standard matrix norms, so taking norms in  
(1.8) it follows that

$$\frac{\|Q_{22}R_{i+1}M_{11}\|}{\|Q_{22}R_iM_{11}\|} \leq \mu + O(\epsilon). \quad (1.9)$$

Since  $Q_{22}$  and  $M_{11}$  are constant and nonsingular, (1.9) implies (1.5) by the equivalence of matrix norms. Also the residual matrix  $R_i$  and error matrix  $E_i$  can be related as

$$\begin{aligned} R_i &= L_i A_{11} - A_{22} L_i - L_i A_{12} L_i + A_{21} \\ &= E_i \tilde{A}_{11} - \tilde{A}_{22} E_i - E_i A_{12} E_i \end{aligned}$$

so that for  $\epsilon$  sufficiently small a decrease in  $R_i$  yields a proportional decrease in  $E_i$ .

The convergence of Algorithm 4.1 might be improved by including a line search in step 3. That is, if  $\|R_{i+1}\| > \|R_i\|$  then let  $L_{i+1} = L_i + hD_i$  and search on  $h$  until  $\|R_{i+1}\|$  is minimized. However, this safeguard procedure was not needed in the numerical examples which follow.

Our procedure for obtaining numerical solutions to the nonsymmetric Riccati equation has some similarities to one proposed by Farrar and DePietro (1977) for solving the symmetric Riccati equation in that both procedures are initialized using eigenvectors. However, their iterative correction involves a bilinear equation whereas Algorithm 4.1 involves a linear iterative correction.

#### 4.2. Computing the K Matrix

Since the Lyapunov equation

$$K\hat{A}_{22} - \hat{A}_{11}K + A_{12} = 0 \quad (2.1)$$

has a unique solution, the choice of solution method is not so important. The two-time-scale structure may again be used to good advantage, as it was in solving for the L matrix in the following

##### Algorithm 4.2.

- 1) Set  $K_0 = 0$ ,  $\hat{R}_0 = -A_{12}$ , and  $i = 0$ .
- 2) Set  $K_{i+1} = K_i + \hat{R}_i \hat{A}_{22}^{-1}$ .
- 3) Evaluate  $\hat{R}_{i+1} = \hat{A}_{11}K_{i+1} - K_{i+1}\hat{A}_{22} - A_{12}$  and stop if  $\|\hat{R}_{i+1}\| \leq \kappa$ .
- 4) Set  $i = i + 1$  and go to step 2.

An advantage of this approach is that  $\hat{A}_{22}^{-1}$  is unchanged in the iteration so that it can be decomposed once into LU form and then used repeatedly in step 2 to find  $\hat{D}_i = K_{i+1} - K_i$ .

The approximate rate of convergence can then be demonstrated as it was for Algorithm 4.1. Thus,

$$\begin{aligned} \hat{R}_{i+1} &= (K_i + \hat{D}_i)\hat{A}_{22} - \hat{A}_{11}(K_i + \hat{D}_i) + A_{12} \\ &= \hat{R}_i + \hat{D}_i\hat{A}_{22} - \hat{A}_{11}\hat{D}_i = -\hat{A}_{11}\hat{R}_i\hat{A}_{22}^{-1} \end{aligned}$$

so

$$\frac{\|R_{i+1}\|}{\|R_i\|} \approx \frac{|s_1|}{|f_{n_1}|} = \mu \ll 1.$$

Hence the algorithm is simple to implement and converges rapidly for systems that are strongly two-time-scale.

#### 4.3. Ordering and Partitioning of the State Variables

In Chapter 2, the vector of state variables,  $x$ , was partitioned into sub-vectors  $x_1$  and  $x_2$  with dimensions determined by the number of slow modes,  $n_1$ . However, there may be more than one possible choice for  $n_1$ . In some cases, all eigenvalues may be well separated in absolute values so any  $1 \leq n_1 < n$  will yield a reasonably small parameter  $\mu$ .

In most cases, the choice of  $n_1$  will be strongly guided by practical considerations. If one is working with systems of large order and is seeking a reduced-order model of the slow mode dynamics, the choice of  $n_1$  would be based on a trade-off between model accuracy and computing time and expense. As described earlier, the number of multiplications per time step for the reduced model increases at least as rapidly as  $n_1^2$ . Also, shorter time steps are required to integrate faster dynamic modes. The advantages of keeping  $n_1$  small are thus quite apparent.

The choice of  $n_1$  might be established by the need to model a given dynamic bandwidth for control system design studies, (e.g., Skira and DeHoff, 1977). If one needs to design a controller to relocate  $n'$  slow open-loop eigenvalues, then provided the two-time-scale assumption is satisfied, any  $n_1 \geq n'$  can be chosen.

After the number  $n_1$  has been chosen the question of which state variables  $x$  should be placed in  $x_1$  arises, i.e., should  $x$  be reordered before it is partitioned?

As previously shown, the only fixed requirement on the ordering of  $x$  is that the resulting submatrix  $M_{11}$  of the modal matrix  $M$  be nonsingular. If the state variables are reordered, then the rows of the modal matrix  $M$  are reordered in the same manner. If the purpose of decoupling is to obtain a reduced order model, this provides no firm constraints on the ordering of  $x$  since the model yields the slow mode dynamics of all state variables. There are, however, sound reasons for considering a reordering of  $x$  before partitioning. If matrices  $L$  and  $K$  can be made small in norm, then both the transformation from state coordinates  $x$  to time scale decoupled coordinates  $y$  defined by  $y = Tx$  and the inverse transformation back to state coordinates  $x = T^{-1}y$  can be performed with minimal numerical error.

#### 4.4. The Relationship Between Ordering and Coupling

The relative size of  $\|M_{11}\|$  and  $\|M_{21}\|$  provides some measure of the distribution of the slow modes between state variables  $x_1$  and  $x_2$ . From our result (1.1) it is apparent that  $\|L\|$  is proportional to  $\|M_{21}\|$  and inversely proportional to  $\|M_{11}\|$ . If one chooses an ordering of  $x$  which minimizes  $\|M_{21}\|$  relative to  $\|M_{11}\|$  this will insure that the slow modes appear primarily in the components of  $x_1$ , that the linear equation (1.1) is well-conditioned, and  $\|L\|$  will be small. In a similar fashion, since

$$K = -M_{12}Q_{22} = -M_{12}(M_{22} - M_{21}M_{11}^{-1}M_{12})^{-1}, \quad (4.1)$$

choosing an ordering of states which minimizes  $\|M_{12}\|$  will approximately place fast variables in  $x_2$  and yield small  $\|K\|$ .

This process is clarified by introducing slow and fast mode coupling ratios  $\rho_s$  and  $\rho_f$ . Assume that the columns of  $M$  are normalized so that each has Euclidean norm one, and define

$$\rho_s \equiv \frac{\|M_{21}\|}{\|M_{11}\|}, \quad \rho_f \equiv \frac{\|M_{12}\|}{\|M_{22}\|}.$$

Minimizing  $\rho_f$  is a much simpler problem than minimizing  $\|K\|$ , cf. (4.1), and has the advantage of specifically

placing states which are predominately fast into  $x_2$ , cf., (II.3.6). Either  $\rho_s$  or  $\rho_f$  can be minimized by searching over all orderings of the row indices  $\{i\}$  of the modal matrix  $M = [m_{ij}]$ .

If  $x$  can be ordered so that  $M_{21} = 0$ , then  $\rho_s = 0$  and the slow modes are naturally decoupled from the  $x_2$  states. For this ordering of  $x$  the  $A$  matrix will be block-triangular. Likewise, if  $M_{12} = 0$ , then  $\rho_f$  is zero and the resulting  $A$  will be block-triangular. Depending upon the structure of  $M$ , the ordering of  $x$  which minimizes  $\rho_s$  may or may not also minimize  $\rho_f$ .

If both  $\rho_s$  and  $\rho_f$  are small we could describe the system as being "weakly coupled". This description of weakly coupled systems is similar to one proposed by Milne (1965). However, his definition also required that  $\mu$  be small. Even though a system is strongly two-time-scale it need not be weakly coupled, or conversely.

The following procedure is suggested for reindexing  $x$ ,  $A$ ,  $B$ ,  $C$ , and  $M$  in order to approximately minimize  $\|L\|$ , place slow variables in  $x_1$ , and avoid a singular  $M_{11}$ . Similar procedures could be adopted in order to approximately minimize either  $\rho_f$  or  $\max\{\rho_s, \rho_f\}$ . It is assumed one is able to compute approximate eigenvectors.

#### Algorithm 4.3.

- 1) Normalize the columns of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \quad (4.10)$$

so that  $\sum_{i=1}^n |m_{ij}|^2 = 1, \quad j = 1, \dots, n_1.$

- 2) If (4.10) contains one or more complex eigenvectors  $v$ , replace  $v$  with  $\text{Re}(v)$  and replace  $\bar{v}$  with  $\text{Im}(v)$  (as described earlier).
- 3) Evaluate the norm of each of the  $n$  rows of the resulting matrix (4.10) and let

$$r_i \equiv \left( \sum_{j=1}^{n_1} m_{ij}^2 \right)^{1/2}, \quad i = 1, \dots, n.$$

- 4) Order the  $\{r_i\}$  so that  $r_{\alpha_1} \geq r_{\alpha_2} \geq \dots \geq r_{\alpha_n}$ , and premultiply  $x$ ,  $A$ ,  $B$ , and  $M$  by

$$\begin{bmatrix} e_{\alpha_1} & e_{\alpha_2} & \dots & e_{\alpha_n} \end{bmatrix}^T$$

where  $e_i$  is the  $i$ th column of the  $n \times n$  identity matrix.

- 5) Post-multiply  $A$  and  $C$  by the inverse matrix

$$\begin{bmatrix} e_{\alpha_1} & e_{\alpha_2} & \dots & e_{\alpha_n} \end{bmatrix}.$$

One advantage to such a systematic approach to approximately minimizing  $\|L\|$  and/or  $\|K\|$  is that it can be readily implemented for systems of large order, where the modal matrix becomes unwieldy for manual manipulations.

#### 4.5. The Computational Method Summarized

To summarize, the proposed procedure for computing the decoupling transformation for a two-time-scale system involves the following steps:

- 1) Obtain approximate eigenvalues  $\lambda(A)$  and choose the number of slow modes  $n_1$ . Note that different choices of  $n_1$  are possible to suit the needs of the study.
- 2) If  $n_1 \leq n_2$  obtain approximate eigenvectors  $\begin{bmatrix} M_{11} \\ M_{12} \end{bmatrix}$ . Otherwise, obtain left eigenvectors  $[Q_{21} \ Q_{22}]$ .
- 3) (Optional.) Apply Algorithm 4.3 to avoid singular  $M_{11}$  and place slow variables in  $x_1$ .
- 4) Eliminate imaginary parts as needed and approximately solve either  $LM_{11} = -M_{21}$  or  $Q_{22}L = Q_{21}$  for  $L = L_0$ .
- 5) (Optional.) Improve the accuracy of  $L_0$  by Algorithm 4.1.
- 6) Find  $K$  by Algorithm 4.2.

Numerical examples of this procedure are presented in the next chapter.

## CHAPTER 5

### NUMERICAL EXAMPLES

Two examples of linear control systems taken from recent literature are presented here to demonstrate actual computation times and convergence rates of the proposed algorithms and also provide further comparison of the new reduced order model with the Marshall model.

The numerical computations were performed at the University of Arizona Computing Center on both the DEC10 and CYBER175 computers to compare the effects of computer word length and compare execution times for time sharing against batch processing.

The performance of Algorithms 4.1 and 4.2 is measured by the parameter  $\log_{10} \|R_i\|$  where  $R_i$  is the appropriate residual error matrix and  $\|\cdot\|$  is the Euclidean norm. As illustrated below, the rate of convergence of these algorithms is well approximated by the system's small parameter,  $\mu$ . The numerical data for these examples is listed in Appendix A.

#### 5.1. A Strongly Two-Time-Scale Example

The first example is a fourth order model of the longitudinal dynamics of an F-8 aircraft used by Teneketzis

and Sandell (1977) to study linear regulator design by a multiple-time-scales method. Their design method is based on the singularly perturbed form of the system described by (II.3.16), and the size of the small parameter was not evaluated. The F-8 aircraft was used by NASA as an experimental vehicle for a digital fly-by-wire technology research program conducted from 1973 to 1978 as described by Elliot (1977). The four state variables for this model are velocity variation  $v$ , flight path angle  $\gamma$ , angle of attack  $\alpha$ , and pitch rate  $\dot{q}$ ; and the one control variable is elevator angle  $u$ . The system eigenvalues are  $s_1, s_2 = -.0075 \pm i.076$  and  $f_1, f_2 = -.94 \pm i2.98$  so the time scales ratio  $|s_2| / |f_1|$  equals .024. Since this value of  $\mu$  is very small relative to one, the system is strongly two-time-scale, and the large difference between speeds of the slow and fast modes is clearly seen in the response to initial conditions as shown in the illustrations which follow.

This system provides an example of highly oscillatory dynamics since the damping ratios for the slow and fast modes are only .1 and .3, respectively. The minimum slow mode coupling ratio  $\rho_s$  is  $2 \times 10^{-4}$  indicating that the slow mode appears predominately in two variables, namely velocity variation and flight path angle. These slow variables are the first two states in vector  $x$ , so state reordering

described in Chapter 4 was not needed, i.e., Algorithm 4.3 was not employed.

The convergence of Algorithms 4.1 and 4.2 for this example is illustrated in Figure 5.1. As shown, the initial matrix  $L_0$  obtained using EISPACK is quite accurate and only two iterations of Algorithm 4.1 are needed to reach the limit accuracy imposed by the finite word length of the computer. The six orders-of-magnitude difference between the minimum residual errors obtained with the DEC10 and the CYBER can be directly attributed to their 36 bit and 64 bit word lengths. Computation times for this example were too short to provide a meaningful comparison, but computation times are later tabulated for the second example.

The new reduced order model and Marshall's reduced order model for the example are compared in Figures 5.2, 5.3 and 5.4. The response to initial conditions for a slow variable, velocity variation, is illustrated in Figure 5.2 and two time-scales are used to show the slow and fast dynamic parts. As indicated in the figure, the new reduced order model provides a more accurate representation of the vehicle response on both the slow and fast time scales except for only the first .05 sec., i.e., the true response equals the value for Marshall's model at time zero and then rapidly departs from that value and approaches the new model values. The response to initial conditions for a fast variable, angle

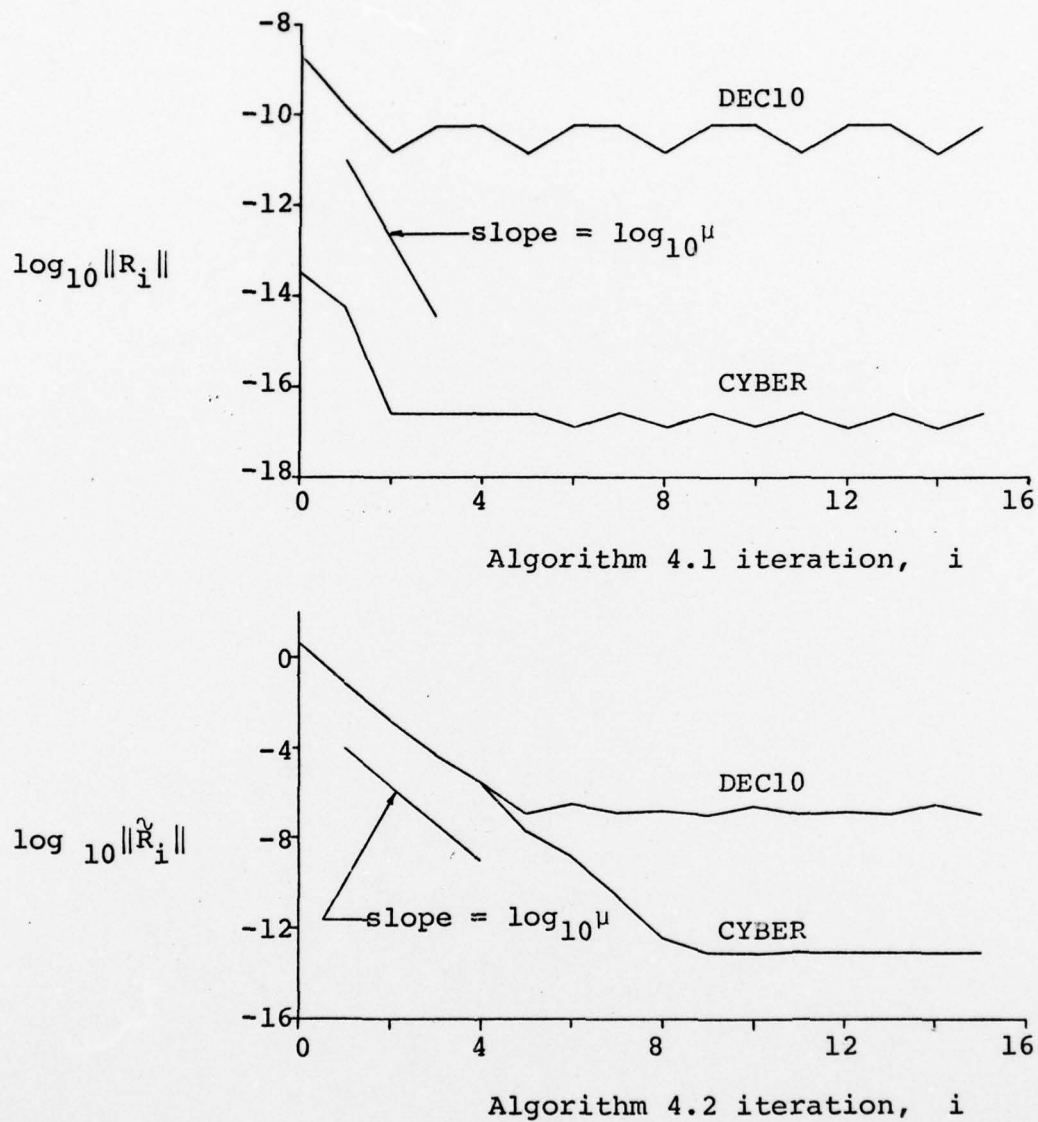


Figure 5.1. Convergence of Algorithms 4.1 and 4.2 for Example 5.1.

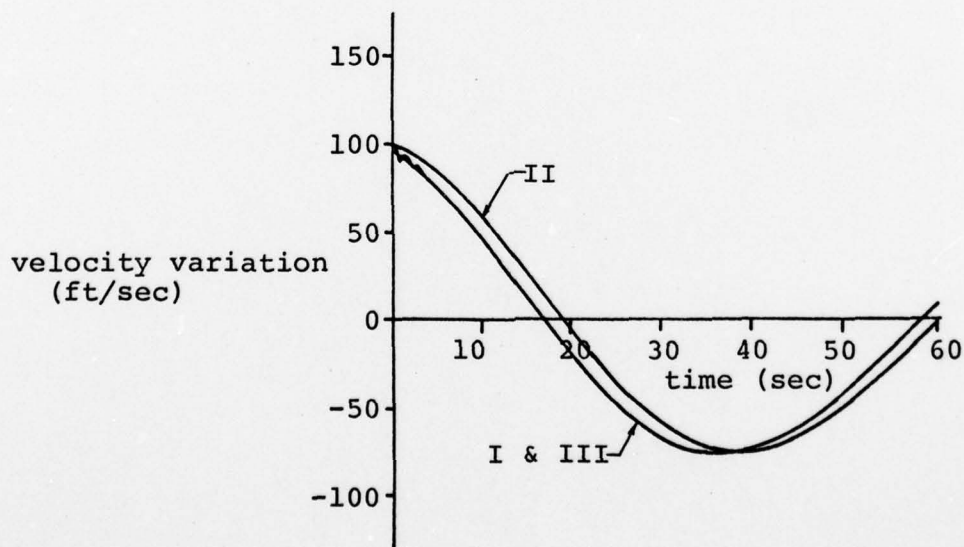
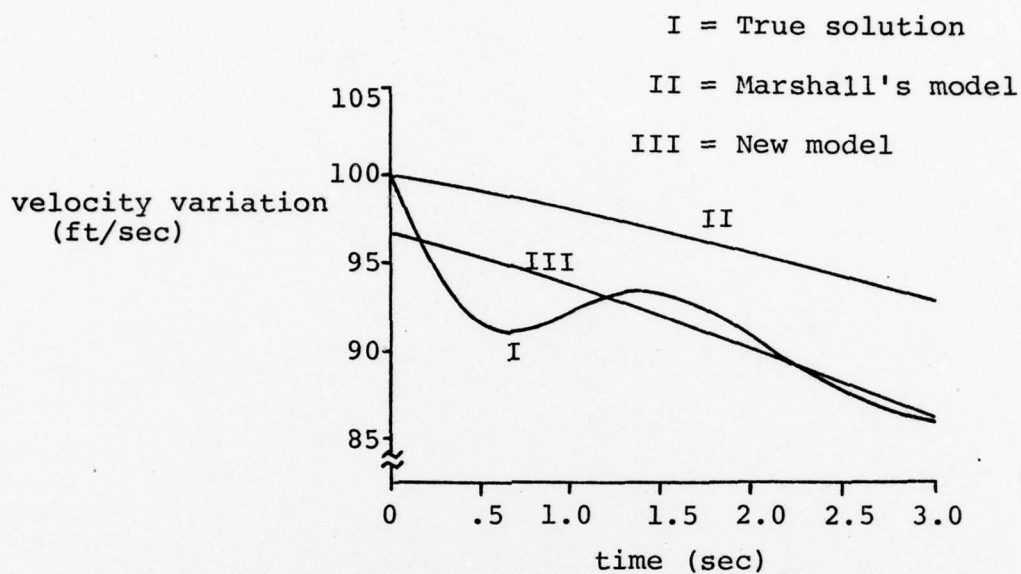


Figure 5.2. Response of a slow variable, velocity variation, of Example 5.1 to initial conditions  $v(0) = 100$  ft/sec,  $\dot{q}(0) = .5$  rad/sec,  $\alpha(0) = 0$ ,  $\gamma(0) = 0$ .

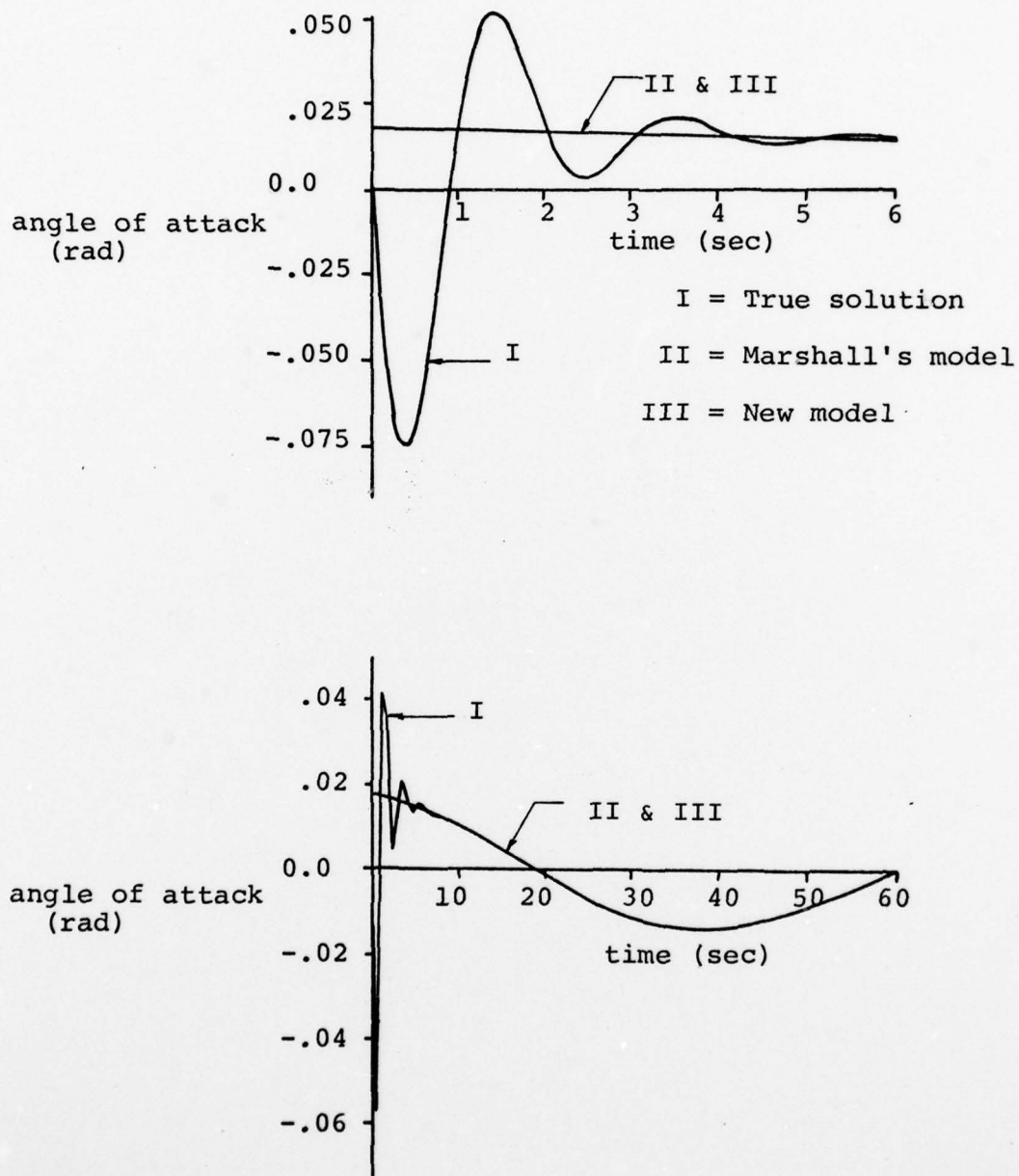


Figure 5.3. Response of a fast variable, angle of attack, of Example 5.1 to initial conditions  $v(0) = 100$  ft/sec,  $\dot{q}(0) = .05$  rad/sec,  $\alpha(0) = 0$ ,  $\gamma(0) = 0$ .

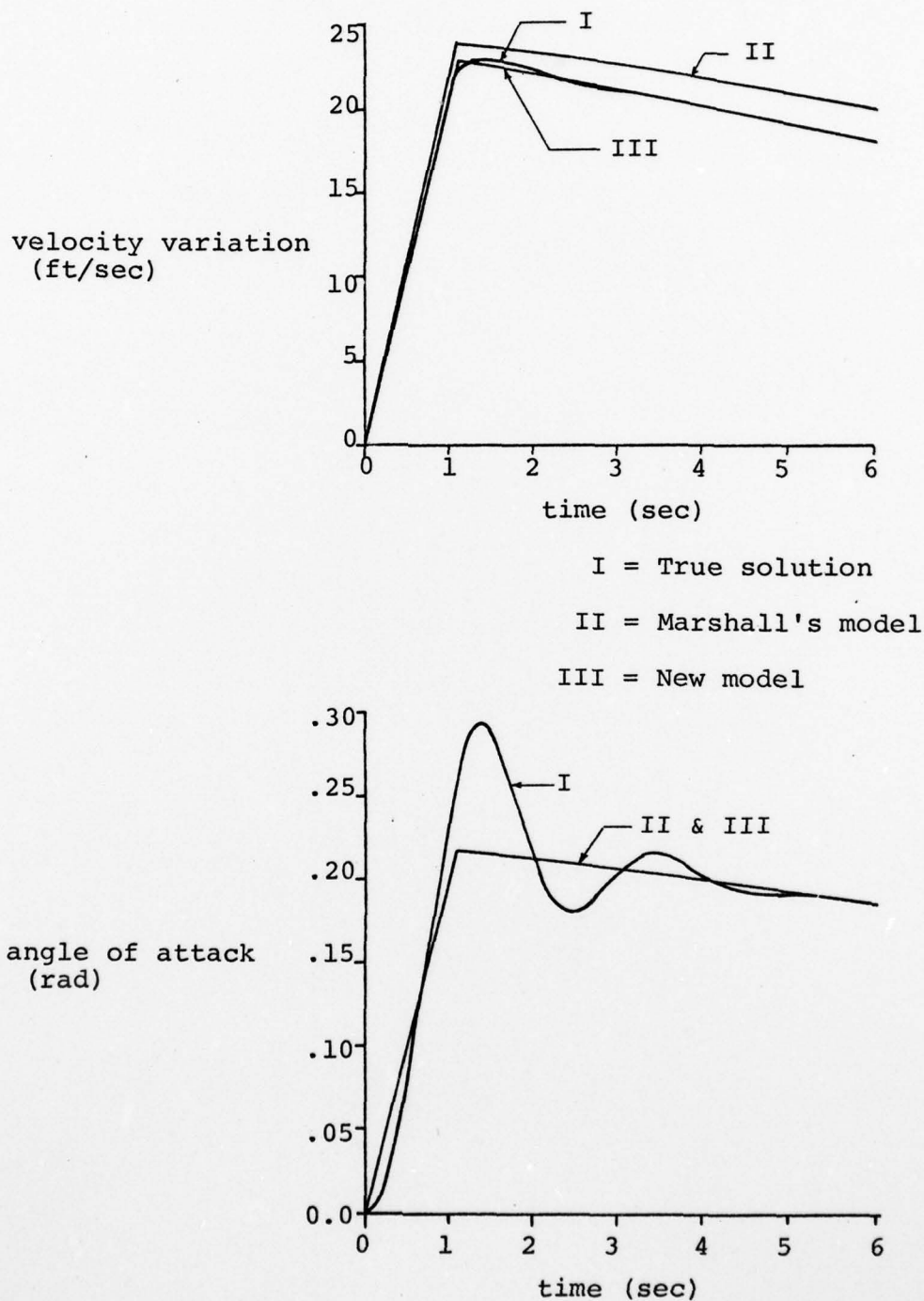


Figure 5.4. Response of Example 5.1 to a .1 (rad) elevator deflection applied for one second.

of attack, is compared in Figure 5.3. For this example both of the reduced order models accurately reproduce the slow component of the fast variable. Since Marshall's model and the new model integrate differential equations with the same eigenvalues, their responses to initial conditions have identical period but differ in phase and amplitude.

The response to control inputs of the two reduced order models and the original model are compared in Figure 5.4. The fact that the new reduced order model correctly decomposes control inputs into slow eigenspace and fast eigenspace parts accounts for the very good match in velocity variation between the true response and the new reduced order model.

#### 5.2. A Weakly Two-Time-Scale Example

This example of a linear control system was selected as the theme problem for the International Forum on Alternatives for Multivariable Control held on October 13 and 14, 1977 in Chicago, cf. Sain (1977). The linear model of an F100 turbofan engine was chosen as a realistic example of a modern control problem involving several control variables which are varied simultaneously and in a coordinated manner to achieve, in this case, the goal of rapid thrust response without violating any of several operating constraints. The engine model is sixteenth order and was obtained from a detailed nonlinear digital simulation of the engine using a

numerical linearization technique described by Hackney, Miller and Small (1977). This linear model represents the turbofan operating at sea level with a power level angle of 83 degrees, i.e., rear maximum non-afterburner power. Actually the linear model represents only one of 36 different linear models corresponding to various altitude, Mach number, and power settings which represent conditions within the flight envelope. The operating point for this linear model represents a condition which every engine must pass through on takeoff.

The sixteen state variables for the turbofan consist of two shaft speeds, three pressures, and eleven temperatures. Five of the state variables can be sensed including two shaft speeds, two pressures, and one temperature. The model includes five control variables of fuel flow, exit nozzle area, two variable vane angles, and compressor bleed; and there are five output variables including net thrust, total engine airflow, temperature, and two stall margins. This model is not used here as a control design problem, but rather to provide a realistic evaluation of Algorithms 4.1 and 4.2.

The eigenvalues for this example are  $-.65$ ,  $-1.90$ ,  $-2.62$ ,  $-6.72 \pm i1.31$ ,  $-17.8 \pm i4.8$ ,  $-18.6$ ,  $-21.3 \pm i0.8$ ,  $-38.7$ ,  $-47.1$ ,  $-50.7$ ,  $-59.2$ ,  $-175.7$ , and  $-577.0$ . If the number of slow modes is chosen as 15, 5 and 3, the resulting small parameters are 0.304, 0.371, and 0.383.

Since these values are not particularly small relative to one, we call this system weakly two-time-scale.

The performance of Algorithms 4.1 and 4.2 for the two cases  $n_1 = 3$  and  $n_1 = 5$  with the state variables in their original order are illustrated in Figures 5.5 and 5.6. As shown, the convergence rate approximately equals the small parameter. Again, the initial matrix  $L_0$  obtained with EISPACK is highly accurate; the limit of numerical accuracy is reached in 6-8 iterations of Algorithm 4.1, and the effects of computer word length are readily apparent. The computation times corresponding to Figures 5.5 and 5.6 are listed in Table 5.1. As expected, the CYBER175 required considerably less central processor time to perform these computations than the DEC10.

The effects of reordering the state variables before partitioning  $x$  into  $(x_1^T, x_2^T)$  are illustrated in Figure 5.7 for the case  $n_1 = 5$ . As shown, the choice of ordering does not significantly affect the rate of convergence of Algorithm 4.1. However, the norms of the  $L$  and  $K$  matrices and the values of  $\rho_s$  and  $\rho_f$  are influenced by the choice of ordering as shown by Table 5.2. Here less accurate eigenvectors were used to calculate  $L_0$  so the rate of convergence of Algorithm 4.1 could be more clearly observed.

The computation of the  $L$  matrix using alternative initial values for  $L_0$  is described in Figure 5.7.

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DECOUPLING AND REDUCED ORDER MODELING OF TWO-TIME-SCALE CONTROL--ETC(U)

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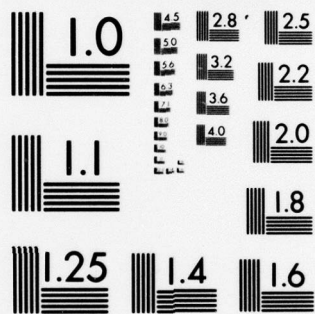
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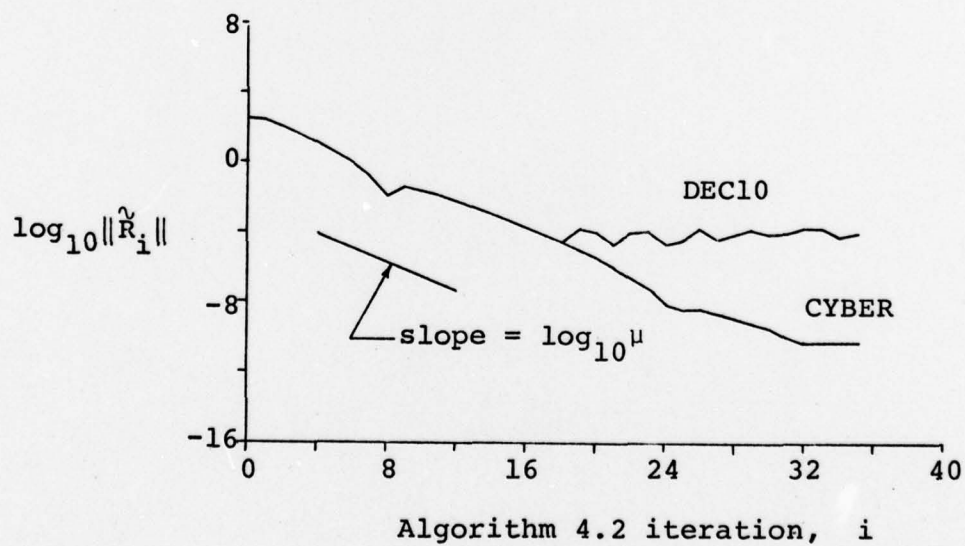
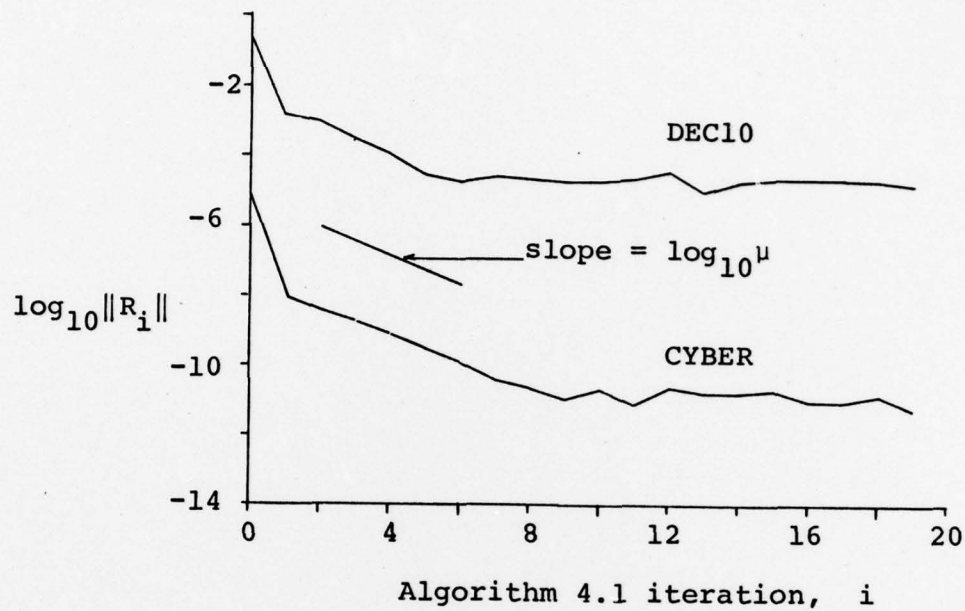


Figure 5.5. Convergence of Algorithm 4.1 and 4.2 for Example 5.2 with  $n_1 = 3$ ,  $\mu = .383$ .

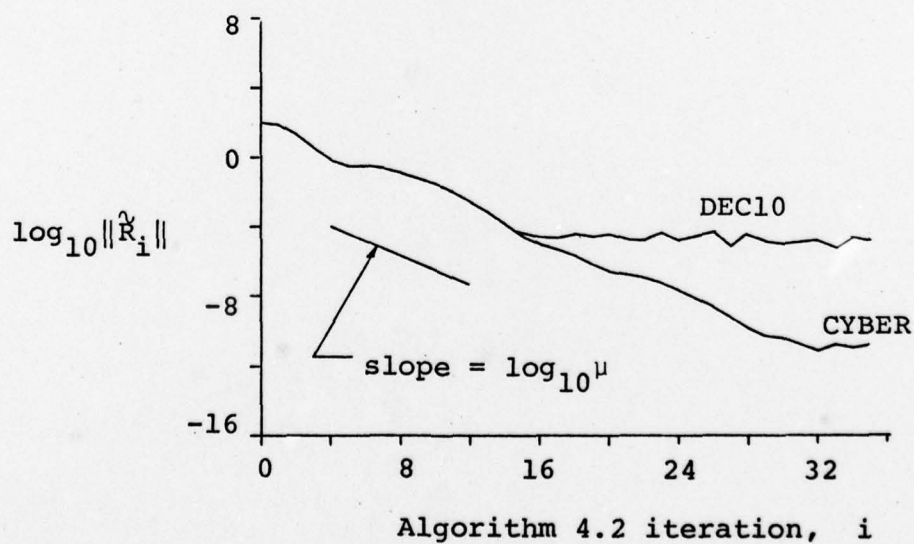
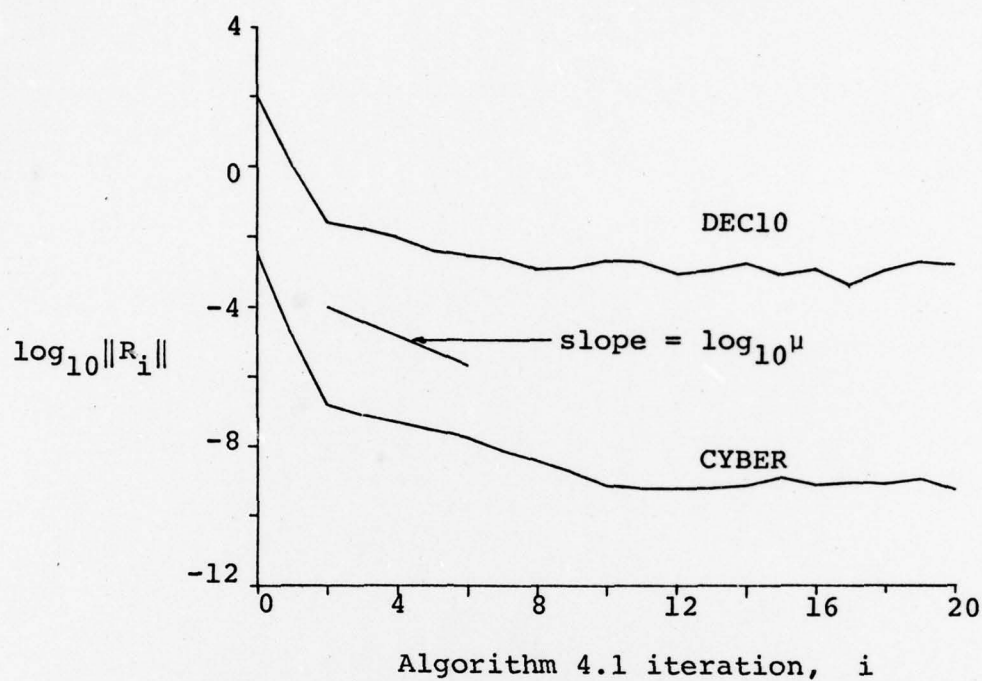


Figure 5.6. Convergence of Algorithms 4.1 and 4.2 for Example 5.2 with  $n_1 = 5$ ,  $\mu = .371$

Table 5.1. Execution Times Corresponding to Figures 5.5 and 5.6.

	Case $n_1 = 5$		Case $n_1 = 3$	
Time to compute $L_0$	.109	.615	.095	.599
Time per $L_i$ iteration (Algorithm 4.1)	.023	.187	.021	.171
Time per $K_i$ iteration (Algorithm 4.2)	.015	.126	.012	.091
Computer	CYBER	DEC10	CYBER	DEC10

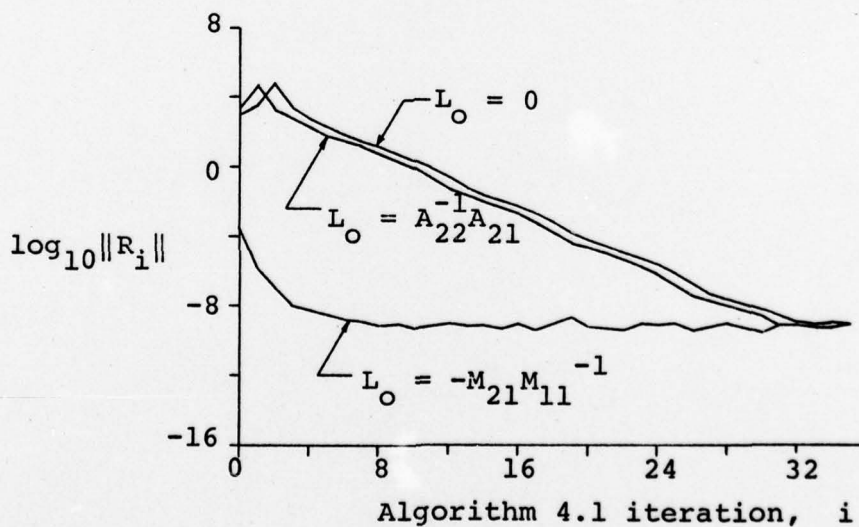
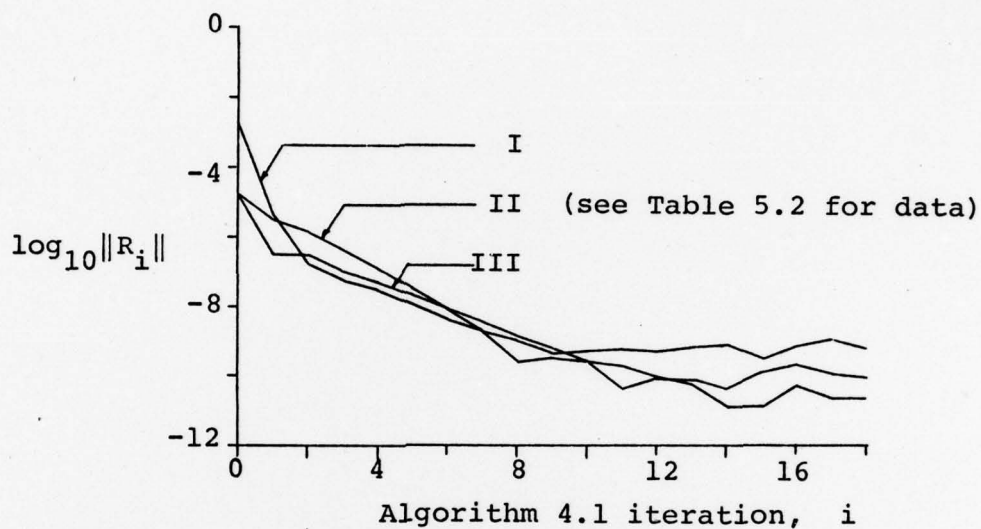


Figure 5.7. Convergence of Algorithm 4.1 for Example 5.2 with  $n_1 = 5$  for alternative ordering of the state and alternative initial  $L_0$ .

Table 5.2. Data for Cases I, II and III.

Case	$x_1$ indices	$\rho_s$	$\ L\ $	$\rho_f$	$\ K\ $
I	1,2,3,4,5	.260	342.	.536	3.42
II	1,2,5,6,10	.248	213.	.340	1.55
III	1,2,9,11,14	.183	39.1	.693	17.1

Case III was obtained by Algorithm 4.3

Algorithm 4.1 is quite stable for this example and succeeds in converging to the correct solution even when initialized with  $L_0 = 0$  or  $L_0 = A_{22}^{-1}A_{21}$ . This last approximation to the  $L$  matrix was proposed by Kokotovic (1975), cf.

(III.2.3a) However, the  $L_0$  matrix obtained using eigenvectors provides a much more reliable initial value for Algorithm 4.1. Nonetheless, good convergence can be expected from eigenvectors which are not highly accurate.

An effort was made to compare the new reduced order model with Marshall's model for this example. However due to the high order and stiffness of this example a reference time history was not obtainable using standard integration methods.

## CHAPTER 6

### EXTENSIONS TO TIME-VARYING

#### CONTROL SYSTEMS

The time-scale decoupling methods presented here may be applied to control systems where all matrices are functions of time. We now assume the control system has the form

$$\dot{x} = A(t)x + B(t)u \quad (0.1)$$

with  $A(t)$ ,  $B(t)$  and  $u(t)$  being specified continuous functions of time on  $[0, T]$  and with the initial state  $x(0)$  specified.

#### 6.1. The Time-Varying LK Transformation

The control system (0.1) will be classified as two-time-scale on the interval  $[0, T]$  if there exist integers  $n_1$  and  $n_2 = n - n_1$  such that the time-varying spectrum  $\lambda(A(t))$  can be partitioned into sets of  $n_1$  and  $n_2$  members

$$\lambda(A(t)) = S(t) \cup F(t) \quad \text{on } [0, T],$$

such that

$$\max_{s_i \in S} |s_i(t)| = \ell(t) \ll v(t) = \min_{f_j \in F} |f_j(t)| \quad (1.1)$$

throughout  $0 \leq t \leq T$  and

$$\mu = \max_{0 \leq t \leq T} \frac{\ell(t)}{v(t)} \ll 1.$$

It is not yet assumed that (0.1) is two-time-scale.

O'Malley (1969) and Harris (1973) have described a time-varying version of the LK transformation for homogeneous singularly perturbed boundary value problems of the form

$$\dot{z}_1 = C_{11}(t)z_1 + C_{12}(t)z_2 \quad (1.3)$$

$$\varepsilon \dot{z}_2 = C_{21}(t)z_1 + C_{22}(t)z_2$$

on the interval  $[0, T]$  where  $z_1$  and  $z_2$  are real vectors of dimensions  $m_1$  and  $m_2$  respectively, and  $C_{ij}(t)$ ,  $i, j = 1, 2$ , are compatible matrix functions of time. The approximate boundary conditions are

$$D_1 z(0) + D_2 z(T) = c \quad (1.4)$$

where  $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ , and  $\varepsilon \ll 1$  is the small parameter. As

shown by O'Malley (1969), if the matrices

$$C_{22}(t) \quad \text{and} \quad C_{11}(t) - C_{12}(t)C_{22}^{-1}(t)C_{21}(t)$$

are nonsingular on  $[0, T]$ , then there exists a nonsingular transformation of variables

$$w = P(t)z$$

which transforms system (1.3) into block-diagonal form

$$\dot{w}_1 = \gamma_{11} w_1$$

$$\epsilon \dot{w}_2 = \gamma_{22} w_2$$

on  $[0, T]$  with boundary conditions (1.4) appropriately transformed. Similarly, Harris (1973) showed that if the eigenvalues of matrix  $C_{22}(t)$  have nonzero real part on  $[0, T]$  then the LK transformation described below exists on  $[0, T]$  and transforms (1.3) to block-diagonal form. Further, the initial value problem has the limiting solution  $w_2 = 0$  if all the eigenvalues of  $\gamma_{22}(t)$  have negative real parts everywhere.

To extend these results to control system (0.1), partition (0.1) using integers  $n_1$  and  $n_2$  as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1(t) \\ B_2(t) \end{bmatrix} u \quad (1.5)$$

where  $x_1$ ,  $x_2$  and  $u$  are vectors of dimensions  $n_1$ ,  $n_2$  and  $m$  respectively, and all matrices are compatible. As for the time-invariant case, the transformation proceeds in two steps. First define new variables  $y_2$  by a nonsingular transformation  $\tilde{L}(t)$

$$\tilde{y} \equiv \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ L(t) & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \equiv \tilde{L}(t)x \quad (1.6)$$

so the transformed system (1.5) becomes

$$\dot{\tilde{y}} = [\tilde{L}A\tilde{L}^{-1} + \dot{\tilde{L}}\tilde{L}^{-1}]\tilde{y} + \tilde{L}Bu \equiv \hat{A}(t)\tilde{y} + \tilde{L}(t)B(t)u. \quad (1.7)$$

The partitions of the  $\hat{A}(t)$  matrix are then

$$\hat{A}_{11} = A_{11} - A_{12}L$$

$$\hat{A}_{12} = A_{12}$$

$$\hat{A}_{21} = \dot{L} - A_{22}L + LA_{11} - LA_{12}L + A_{21}$$

and

$$\hat{A}_{22} = A_{22} + LA_{12}$$

where all matrices are functions of time. In order for (1.7) to be upper block triangular, we need  $\hat{A}_{21}(t) \equiv 0$ , so the  $L(t)$  matrix must satisfy the Riccati differential equation

$$\dot{L} = A_{22}(t)L - LA_{11}(t) + LA_{12}(t)L - A_{21}(t) \quad (1.8)$$

on  $[0, T]$ . Note, however, that from the definition of  $\hat{A}$  if  $\dot{L}(t) \neq 0$  at some point  $t' \in [0, T]$  then  $\hat{A}(t')$  need not be similar to  $A(t')$  and the eigenvalues of the block-triangular system need not necessarily equal those of the original system.

For the second step of the transformation define new variables  $y_1$  by the nonsingular transformation

$$y \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I & K(t) \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} \equiv \tilde{K}(t)\tilde{y} = \tilde{K}(t)\tilde{L}(t)x \equiv T(t)x \quad (1.9)$$

where

$$T(t) = \begin{bmatrix} I + K(t)L(t) & K(t) \\ L(t) & I \end{bmatrix}, \quad (1.10)$$

so the system (1.7) is transformed to

$$\dot{y} = [TAT^{-1} + \dot{T}T^{-1}]y + TBu \equiv \tilde{A}(t)y + \tilde{B}u \quad (1.11)$$

where

$$\tilde{A}_{11} = \hat{A}_{11} = A_{11} - A_{12}L$$

$$\tilde{A}_{12} = \dot{K} + K\hat{A}_{22} - \hat{A}_{11}K + A_{12}$$

$$\tilde{A}_{21} = 0$$

and

$$\tilde{A}_{22} = \hat{A}_{22} = A_{22} + LA_{12}.$$

Now  $\tilde{A}$  will be block-diagonal on  $[0, T]$  if the  $K(t)$  matrix satisfies the linear differential equation

$$\dot{K} = \tilde{A}_{11}(t)K - K\tilde{A}_{22}(t) - A_{12}(t) \quad (1.12)$$

on  $[0, T]$ , and the original system (1.5) is reduced to the decoupled subsystems (1.11), i.e.,

$$\dot{y}_1 = \hat{A}_{11}y_1 + \hat{B}_1u \quad (1.13a)$$

$$\dot{y}_2 = \hat{A}_{22}y_2 + \hat{B}_2u \quad (1.13b)$$

where  $\hat{B}$  is partitioned compatibly into  $\begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}$ . The initial conditions appropriate for (1.13) are

$$y(0) = T(0)x(0).$$

Provided  $L(t)$  and  $K(t)$  satisfy (1.8) and (1.12), the solution  $x(t)$  to the original system may be obtained by integrating (1.13) and transforming  $y(t)$  as

$$x(t) = T(t)^{-1}y(t) = \begin{bmatrix} I & -K(t) \\ -L(t) & I + L(t)K(t) \end{bmatrix} y(t).$$

Systems of linear differential equations related by a non-singular linear transformation such as (1.9) are said to be kinematically similar (Coppel, 1978).

In order for the transformation (1.9) and thereby system (1.13) to be completely specified, appropriate boundary values for matrices  $L(t)$  and  $K(t)$  need to be chosen. At this point we will assume that the decoupled system (1.13)

is two-time-scale. Then proceeding as in the time invariant case,  $L(0)$  can be chosen to initially separate time scales between  $\hat{A}_{11}$  and  $\hat{A}_{22}$  so that

$$\lambda(A_{11}(0) - A_{12}(0)L(0)) = \hat{S}(0) = S(0) \quad (1.14a)$$

and

$$\lambda(A_{22}(0) + L(0)A_{12}(0)) = \hat{F}(0) = F(0) \quad (1.14b)$$

where  $\lambda(\hat{A}(0)) = \hat{S}(0) \cup \hat{F}(0)$ , and

$$\dot{L}(0) = 0. \quad (1.15)$$

(Then, we guarantee that the eigenvalues of  $A(0)$  are preserved in  $\hat{A}(0)$ , i.e., the initial time-scale-decoupling is preserved under the transformation.) Let  $A(0)$  have the spectral form

$$A(0) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where  $J_1$  and  $J_2$  are Jordan matrices with

$$\lambda(J_1) = S(0), \quad \lambda(J_2) = F(0)$$

and  $M^{-1} = Q$ . Then assuming  $M_{11}$  is nonsingular (rearranging rows of  $x$ , if necessary)  $L(0)$  will satisfy

$$L(0) = -M_{21}M_{11}^{-1} \quad (1.16)$$

and (1.14) and (1.15) will both be satisfied. Provided

$$T(t)B(t)u(t) = \hat{B}(t)u(t)$$

is slowly varying on  $[0, T]$ , this choice of initial conditions on  $L$  will yield separation of fast and slow dynamics in (1.13) at the initial time  $t = 0$  and, by assumption, this separation of time scales in (1.13) persists throughout  $[0, T]$ . If this fails to persist, our procedure will not be applicable.

One would hope to be able to solve for  $L(t)$  and  $K(t)$  using some standard differential equation integrator in a numerically stable manner. To assess the stability of (1.8) assume that a matrix  $L(t)$  satisfies (1.8) with initial condition (1.16) and let some other solution  $\hat{L}(t)$  satisfying (1.8) be displaced from  $L(t)$  at time  $t' \in (0, T)$  by some small amount. Define the perturbation matrix  $E(t)$  to be

$$E(t) = \hat{L}(t) - L(t)$$

for  $t \in [t', T]$  where  $E(t')$  is  $O(\delta)$  for some small number  $\delta > 0$ . Then the perturbation matrix satisfies

$$\begin{aligned}\dot{E} &= A_{22}(\hat{L} - L) - (\hat{L} - L)A_{11} + \hat{L}A_{12}\hat{L} - LA_{12}L \\ &= \hat{A}_{22}E - E\hat{A}_{11} + EA_{12}E\end{aligned}\quad (1.17)$$

where  $\hat{A}_{11} = A_{11} - A_{12}L$  and  $\hat{A}_{22} = A_{22} + LA_{12}$ . Since it is assumed that the decoupled system (1.13) is two-time-scale and  $EA_{12}E$  is initially  $O(\delta^2)$ , the perturbation equation (1.17) can be approximated by

$$\dot{E} \approx \hat{A}_{22}E$$

and the stability of (1.17) will depend upon the stability of the fast subsystem (1.13b). Thus to insure the stability of the Riccati differential equation (1.8), it is assumed that the fast subsystem (1.13b) is stable on  $[0, T]$ . Sufficient conditions for the stability of time-varying linear systems (e.g., Davison 1968) are not as simple as those for constant linear systems, and our previous conditions (II.3.1) for fast mode stability are no longer generally adequate.

For  $\mu$  sufficiently small, however, the singular perturbation results cited show that fast mode stability is

guaranteed provided the large eigenvalues of  $\hat{A}_{22}$  have large negative real parts. The numerical stability of the linear differential equation (1.12) may be assessed by a similar argument. Let matrices  $K(t)$  and  $\hat{K}(t)$  both satisfy (1.12) and define the perturbation matrix

$$G(t) = \hat{K}(t) - K(t).$$

Then

$$\dot{G} = \hat{A}_{11}G - G\hat{A}_{22} \quad (1.18)$$

and by the assumption that  $\hat{A}$  is two-time-scale (1.18) can be approximated as

$$\dot{G} \cong -G\hat{A}_{22}.$$

If the fast subsystem (1.13b) is stable,  $K(t)$  can be expected to grow without bound as (1.12) is integrated forward in time. This is demonstrated in the numerical example which follows.

However, the linear differential equation (1.12) will be stable when integrated backward in time from  $T$  to 0 and one can obtain  $K(t)$  in this fashion by choosing an

appropriate final time value  $K(T)$ . One natural choice for  $K(T)$  is the unique solution to the linear equation

$$0 = \tilde{A}_{11}(T)K - K\tilde{A}_{22}(T) - A_{12}(T)$$

obtained by the iteration

$$K = (\tilde{A}_{11}(T)K - A_{12}(T))\tilde{A}_{22}^{-1}(T)$$

so that

$$\dot{K}(T) = 0.$$

Based on these findings, the following procedure for decoupling an initially two-time-scale linear control system (1.5) is suggested.

- 1) At the initial time  $t = 0$ , apply the methods of the previous chapters to find the unique matrix  $L(0)$  satisfying

$$0 = A_{22}(0)L(0) - L(0)A_{11}(0) + L(0)A_{12}(0)L(0) - A_{21}(0)$$

with

$$\lambda(A_{11}(0) - A_{12}(0)L(0)) = \lambda(S(0)).$$

- 2) Integrate the time-varying Riccati differential equation

$$\dot{L} = A_{22}(t)L - LA_{11}(t) + LA_{12}(t)L - A_{21}(t)$$

from  $t = 0$  to  $t = T$  and save the solution  $L(t)$ .

(We assume that  $L(t)$  remains bounded throughout the interval.)

- 3) At the final time  $t = T$ , find the unique matrix  $K(T)$  which satisfies the algebraic equation

$$K(T) = (\tilde{A}_{11}(T)K(T) - A_{12}(T))\tilde{A}_{22}^{-1}(T).$$

- 4) Integrate the time-varying linear differential equation

$$\dot{K} = \tilde{A}_{11}(t)K - K\tilde{A}_{22}(t) - A_{12}(t)$$

backward in time from  $t = T$  to  $t = 0$  and save the solution  $K(t)$ .

- 5) Using  $L(t)$  and  $K(t)$ , integrate the slow and fast mode decoupled subsystems

$$\dot{Y}_1 = (A_{11} - A_{12}L)Y_1 + [(I + KL)B_1 + KB_2]u$$

$$\dot{Y}_2 = (A_{22} + LA_{12})Y_2 + (LB_1 + B_2)u$$

with initial conditions

$$y_1(0) = (I + K(0)L(0))x_1(0) + K(0)x_2(0)$$

$$y_2(0) = L(0)x_1(0) + x_2(0),$$

and transform back to obtain values  $x(t)$  as needed, where

$$x(t) = \begin{bmatrix} I \\ -L(t) \end{bmatrix} y_1(t) + \begin{bmatrix} -K(t) \\ I + L(t)K(t) \end{bmatrix} y_2(t).$$

As an alternative to 5), because the fast mode subsystem (1.13b) is assumed to be stable on  $[0, T]$  and a reduced order approximation  $\hat{x}(t)$  to the time solution  $x(t)$  is desired, the fast subsystem variables  $y_2(t)$  could be approximated by the algebraic relationship

$$\hat{y}_2(t) \equiv -(A_{22} + LA_{12})^{-1}(LB_1 + B_2)u(t).$$

Then the approximate state  $\hat{x}(t)$  is given by

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{I} \\ -\mathbf{L}(t) \end{bmatrix} \mathbf{y}_1(t) + \begin{bmatrix} -\mathbf{K}(t) \\ \mathbf{I} + \mathbf{L}(t)\mathbf{K}(t) \end{bmatrix} \hat{\mathbf{y}}_2(t)$$

with  $\mathbf{y}_1$  again determined by numerical integration.

### 6.2. A Second-Order Example

The decoupling procedure described above is illustrated with the following second-order example.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 + 3h(t) & -6 + h(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (2.1)$$

for  $t \in [0, 5]$  where

$$h(t) = \begin{cases} \sin(\frac{\pi t}{2}), & 0 \leq t \leq 4 \\ 0, & 4 < t \leq 5 \end{cases}$$

with initial conditions

$$x_1(0) = 1, \quad x_2(0) = 1$$

and control input

$$u(t) = \begin{cases} \sin^2(\frac{\pi t}{4}), & 0 \leq t \leq 4 \\ 0 & , \quad 4 < t \leq 5. \end{cases}$$

The initial eigenvalues for this example are  $\lambda_1, \lambda_2 = -1, -5$ . They vary with time as shown in Figure 6.1.

The Riccati differential equation for this example is the scalar equation

$$\dot{L} = (-6 + h(t))L + L^2 - (-5 + 3h(t))$$

with the initial condition

$$L(0) = 1$$

providing  $\dot{L}(0) = 0$ . The  $L(t)$  time history from  $t = 0$  to  $t = 5$  is illustrated in Figure 6.2, and the corresponding fast and slow system matrices

$$\tilde{A}_{11} = A_{11} - A_{12}L, \quad \tilde{A}_{22} = A_{22} + LA_{12}$$

are shown in Figure 6.1 along with the original system's eigenvalues  $\lambda_1$  and  $\lambda_2$ . As shown,  $\tilde{A}_{11}$  and  $\tilde{A}_{22}$  closely follow  $\lambda_1$  and  $\lambda_2$  here.

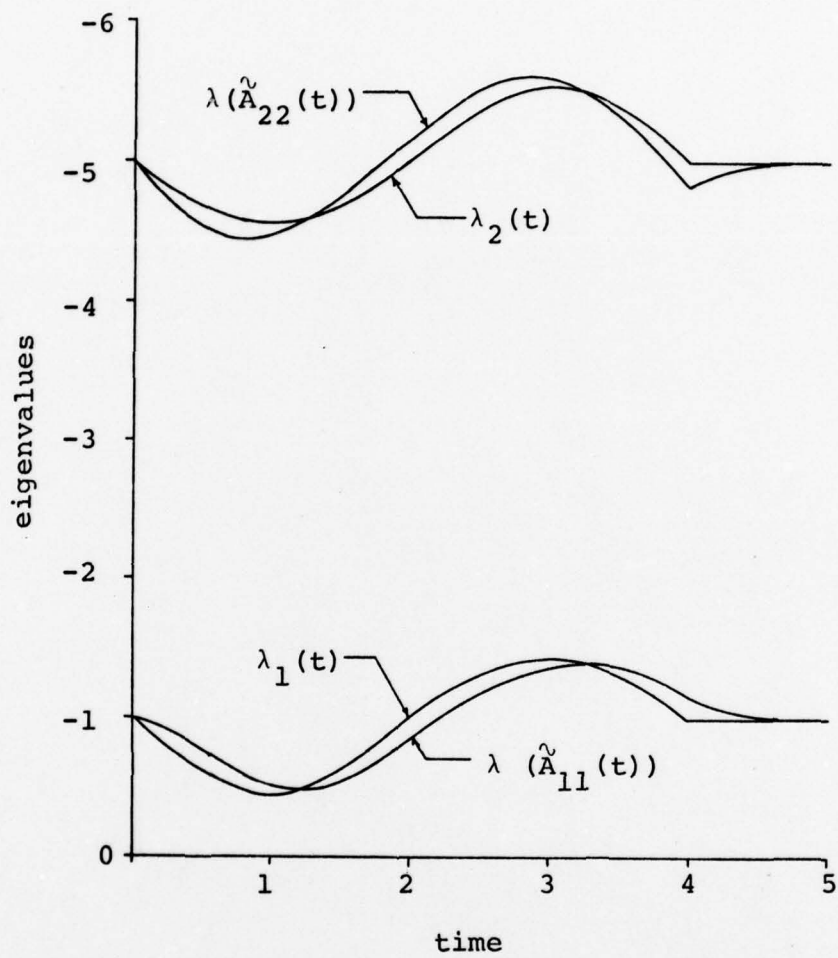


Figure 6.1. Eigenvalues of original and block-decoupled system for Example 6.2.

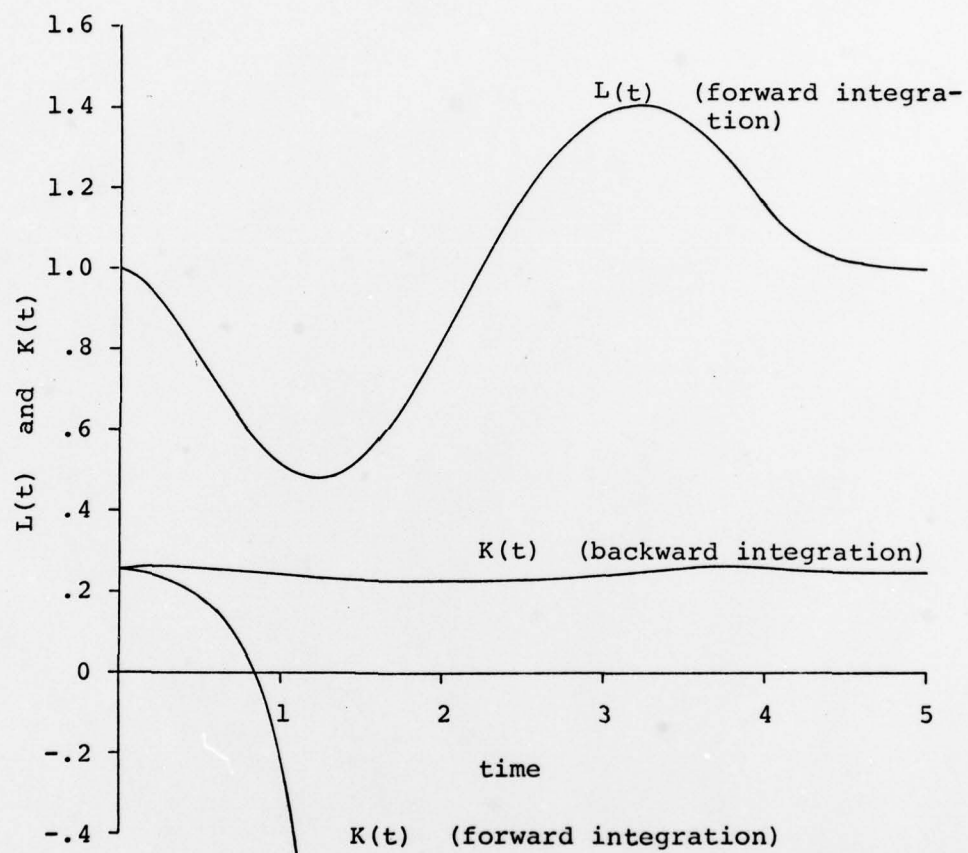


Figure 6.2.  $L(t)$  and  $K(t)$  time histories for Example 6.2.

The scalar differential equation for  $K(t)$  to be integrated backward is

$$\dot{K} = -LK + K(-6 + h + L) - (-5 + 3h)$$

with final value

$$K(5) = \frac{1}{4}$$

corresponding to  $L(5) \approx 1$ . The  $K(t)$  trajectory is illustrated in Figure 6.2. In order to demonstrate the instability of the differential equation for  $K(t)$  when integrated forward in time, a small portion of such a trajectory is also shown in Figure 6.2.

Since the system is linear, the superposition principle applies, and the solution  $x(t)$  can be considered as the sum of two parts; the response to initial conditions  $x(0) = (1, 1)^T$  with zero control input  $u(t)$ , and the response to control input  $u(t)$  with zero initial state. Figure 6.3 illustrates the response of the reconstructed states  $x_1(t)$ ,  $x_2(t)$  and reduced-order approximations  $\hat{x}_1(t)$ ,  $\hat{x}_2(t)$  to initial conditions, whereas Figure 6.4 illustrates the response of these variables to the control input. As shown, the state  $\hat{x}$  provides a reasonably good approximation to  $x$  for this example away from an initial

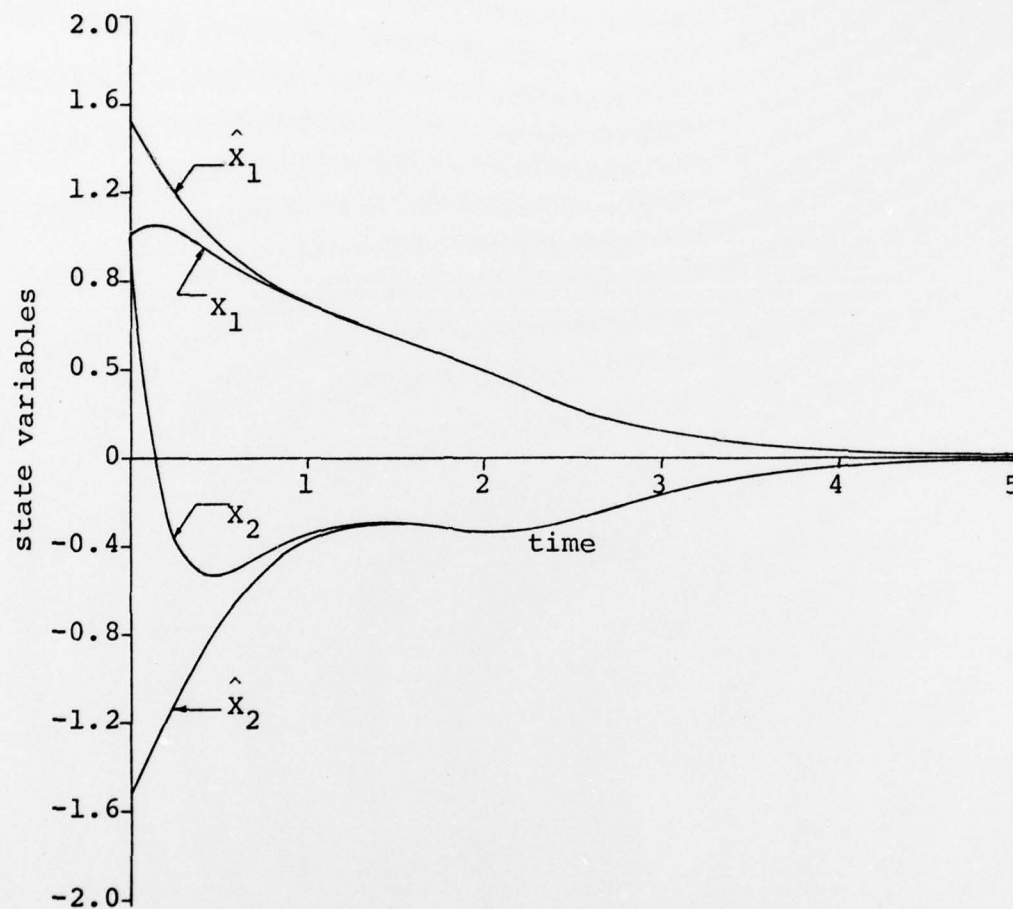


Figure 6.3. Response of state variables to initial conditions for Example 6.2.

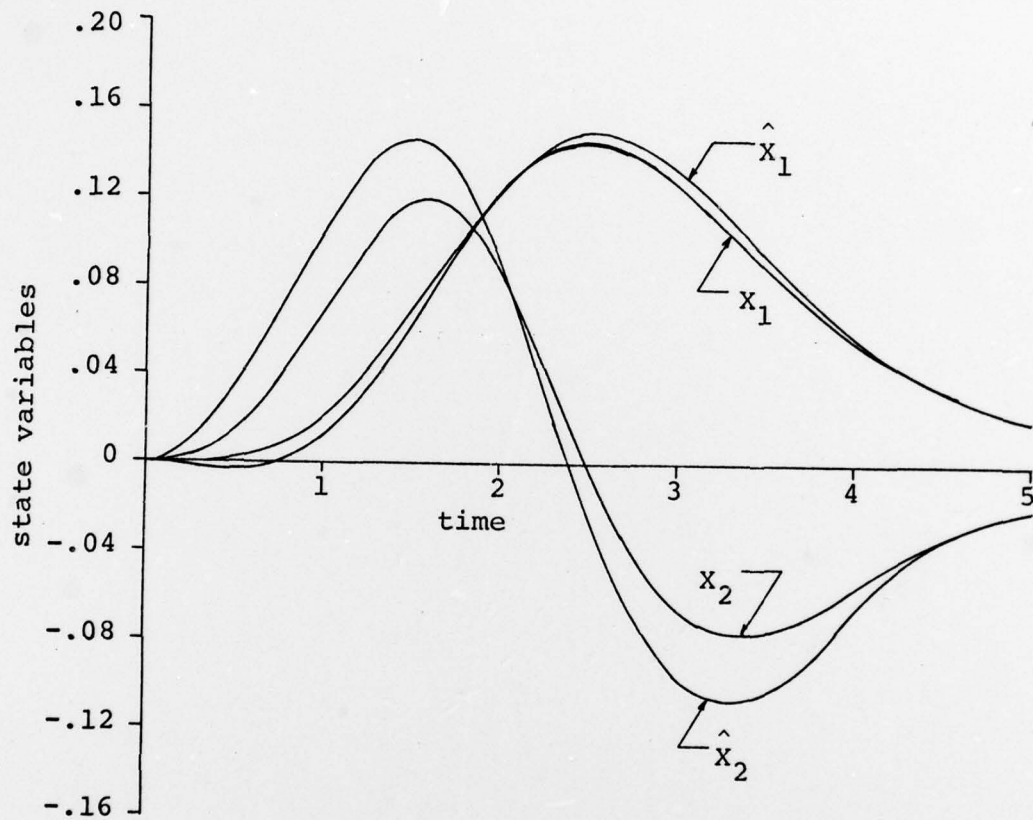


Figure 6.4. Response of state variables to control inputs for Example 6.2.

boundary layer where initial conditions are satisfied. The example was also integrated in its original form (2.1) to verify that the reconstructed state obtained as  $x = \tau^{-1}y$  is correct.

As demonstrated by this example, the time scale decoupling transformation can be extended to time varying linear control systems in a straightforward manner. The method should prove to be particularly useful for large scale linear systems where

- 1) the  $L(t)$  and  $K(t)$  matrices are slowly time varying, and
- 2) the decoupled system is strongly two-time-scale.

In this case only the fast subsystem would require short integration steps. Also, for strongly two-time-scale systems where the fast subsystem is strongly stable, the reduced-order approximation proposed here could lead to considerable reduction in computation time requirements, especially for large order problems.

## APPENDIX A

### DATA FOR EXAMPLES 5.1 AND 5.2

The state and control variables for the aircraft longitudinal dynamics example are

$$x = \begin{bmatrix} v \\ \gamma \\ \alpha \\ q \end{bmatrix}, \quad u = \delta_e$$

where

$v$  = horizontal-velocity deviation (ft/sec)

$\gamma$  = flight-path angle (radians)

$\alpha$  = angle of attack (radians)

$q$  = pitch rate (radians/sec)

$\delta_e$  = elevator deflection (radians).

The corresponding system matrices are

$$A = \begin{bmatrix} -.01357 & -32.2 & -46.3 & 0.0 \\ .0001200 & 0.0 & 1.214 & 0.0 \\ -.0001212 & 0.0 & -1.214 & 1.0 \\ .0005700 & 0.0 & -9.01 & -.6696 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -.4330 \\ .1394 \\ -.1394 \\ -.1577 \end{bmatrix}.$$

The sixteen state variables for the turbofan engine example are

- $x_1$  = fan speed (rpm)
- $x_2$  = compressor speed (rpm)
- $x_3$  = compressor discharge pressure (psia)
- $x_4$  = interturbine volume pressure (psia)
- $x_5$  = augmentor pressure (psia)
- $x_6$  = fan inside diameter discharge temperature ( $^{\circ}$ R)
- $x_7$  = duct temperature ( $^{\circ}$ R)
- $x_8$  = compressor discharge temperature ( $^{\circ}$ R)
- $x_9$  = burner exit fast response temperature ( $^{\circ}$ R)
- $x_{10}$  = burner exit slow response temperature ( $^{\circ}$ R)
- $x_{11}$  = burner exit total temperature ( $^{\circ}$ R)
- $x_{12}$  = fan turbine inlet fast response temperature ( $^{\circ}$ R)
- $x_{13}$  = fan turbine inlet slow response temperature ( $^{\circ}$ R)
- $x_{14}$  = fan turbine exit temperature ( $^{\circ}$ R)
- $x_{15}$  = duct exit temperature ( $^{\circ}$ R)
- $x_{16}$  = augmentor temperature ( $^{\circ}$ R)

and the five engine input variables are

$u_1$  = main burner fuel flow (lb/hr)

$u_2$  = nozzle jet area (ft<sup>2</sup>)

$u_3$  = inlet guide vane position (deg)

$u_4$  = high variable stator position (deg)

$u_5$  = customer compressor bleed flow (%).

The numerical data for the A and B matrices for the turbofan engine are listed in Tables A.1 and A.2.

Table A.1. The A Matrix for the Turbofan Engine Model.

Columns 1 through 8:

-4.328	.1714	5.376	401.6	-724.6	-1.933	1.020	-.9820
-.4402	-5.643	127.5	-233.5	-434.3	26.59	2.040	-2.592
1.038	6.073	-165.0	-4.483	1049.	-82.45	-5.314	5.097
.5304	-.1086	131.3	-578.3	102.0	-9.240	-1.146	-2.408
.8476E-02	-.1563E-01	.5602E-01	1.573	-10.05	.1952	-.8804E-02	-.2110E-01
.8350	-.1249E-01	-.3567E-01	-.6074	37.65	-19.79	-.1813	-.2962E-01
.6768	-.1264E-01	-.9683E-01	-.3567	80.24	-.8239E-01	-20.47	-.3928E-01
-.9696E-01	.8666	16.87	1.051	-102.3	29.66	.5943	-19.97
-.8785E-02	-.1636E-01	.1847	.2169	-8.420	.7003	.5666E-01	6.623
-.1298E-03	-.2430E-03	.2718E-02	.3214E-02	-.1246	.1039E-01	.8395E-03	.9812E-01
-1.207	-6.717	26.26	12.49	-1269.	103.0	7.480	36.84
-.2730E-01	-.4539	-52.72	198.8	-28.09	2.243	.1794	9.750
-.1206E-02	-.2017E-01	-2.343	8.835	-1.248	.9975E-01	.8059E-02	.4333
-.1613	-.2469	-24.05	23.38	146.3	1.638	.1385	4.486
-.1244E-01	.3020E-01	-.1198	-.4821E-01	5.675	-.4525	19.81	.1249
-1.653	1.831	-3.822	113.4	341.4	-27.34	-2.040	-.6166

Columns 9 through 16:

.9990	1.521	-4.062	9.567	10.08	-.6017	-.1312	.9602E-01
11.32	10.90	-4.071	-.5739E-01	-.6063	-.7488E-01	-.5936	-.9602E-01
-.9389E-02	.1352	5.638	.2246E-01	.1797	.2407E-01	1.100	.2743E-01
-3.081	-4.529	5.707	-.2346	-2.111	-.2460	-.4686	-.3223
.2090E-02	-.5256E-01	-.4077E-01	-.9182E-02	-.8178E-01	.3428E-01	.4995E-02	-.1256E-01
-.1953E-01	-.1622	-.6439E-02	-.2346E-01	-.2201	-.2514E-01	-.3749E-02	-.3361E-01
.1878E-01	-.2129	-.9337E-02	-.3144E-01	-.2919	-.3370E-01	.8873E-01	-.4458E-01
.2253E-01	.1791	.8371E-02	.2645E-01	.2560	.2835E-01	-.3749E-01	.3635E-01
-49.99	.6760E-01	39.46	.4991E-02	.8983E-01	.5349E-02	.0	.1372E-01
-.6666	-.6657	.5847	.6654E-04	.1347E-02	.7131E-04	.0	.2057E-03
.2854	2.332	-47.65	.3406	3.065	.3624	-.4343	.4681
-9.627	-9.557	38.48	-50.01	.1011	.1203E-01	-.4686E-01	.1715E-01
-.4278	-.4245	1.710	-2.000	-1.996	.5349E-03	-.1999E-02	.7544E-03
-4.414	-4.354	17.66	-3.113	-3.018	-19.77	-.4999E-01	.1509E-01
-.1127E-02	-.6760E-02	.1835E-01	-.9981E-03	-.1347E-01	-.1070E-02	-20.00	-.2057E-02
.5004	-.1437	-2.416	-.1073	-1.078	30.63	19.89	-50.16

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Table A.2. The B Matrix for the Turbofan Engine Model.

-.4570E-01	-451.6	-105.8	-1.506	851.5
.1114	-546.1	-6.575	-107.8	3526.
.2153	1362.	13.46	20.14	-.6777E+05
.3262	208.0	-2.888	-1.653	-269.1
.9948E-02	-98.39	.5069	-.1940	-94.70
.2728E-01	71.62	9.608	-.3160	-184.1
.1716E-01	71.71	8.571	.7989	-515.2
-.7741E-01	-141.2	-.8215	39.74	1376.
.3855E-01	-7.710	-.4371E-01	-.1024	-6684.
.5707E-03	-.1144	-.6359E-03	-.1432E-02	-99.02
5.727	-1745.	-8.940	-17.96	.8898E+05
.1392	-24.30	-.2736	-.3403	-6931.
.6172E-02	-1.082	-.1183E-01	-.1452E-01	-307.7
.6777E-01	16.60	.3980	.2311E-01	-2588.
.1880E-02	9.147	-.8241	.8984E-01	-32.31
.1677	435.8	-89.94	4.900	-295.5

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